Uncertainty quantification in Structural Dynamics: Day 1

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Education:

- PhD (Engineering), 2001, University of Cambridge (Trinity College), Cambridge, UK.
- MSc (Structural Engineering), 1997, Indian Institute of Science, Bangalore, India.
- B. Eng, (Civil Engineering), 1995, Calcutta University, India.

Work:

- 04/2007-Present: Professor of Aerospace Engineering, Swansea University (Civil and Computational Engineering Research Centre).
- 01/2003-03/2007: Lecturer in dynamics: Department of Aerospace Engineering, University of Bristol.
- 11/2000-12/2002: Research Associate, Cambridge University Engineering Department (Junior Research Fellow, Fitzwilliam College, Cambridge) .

The course is dived into eight topics:

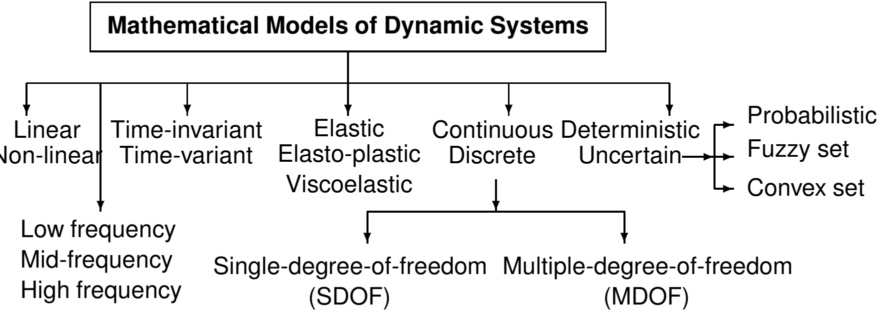
- Introduction to probabilistic models & dynamic systems
- Stochastic finite element formulation
- Numerical methods for uncertainty propagation
- Spectral function method
- Parametric sensitivity of eigensolutions
- Random eigenvalue problem in structural dynamics
- Random matrix theory - formulation
- Random matrix theory - application and validation

1. **Introduction**
2. **Linear dynamic systems**
   - Undamped systems
   - Proportionally damped systems
3. **Random variables**
4. **Random fields**
5. **Stochastic single degrees of freedom system**
6. **Stochastic finite element formulation**

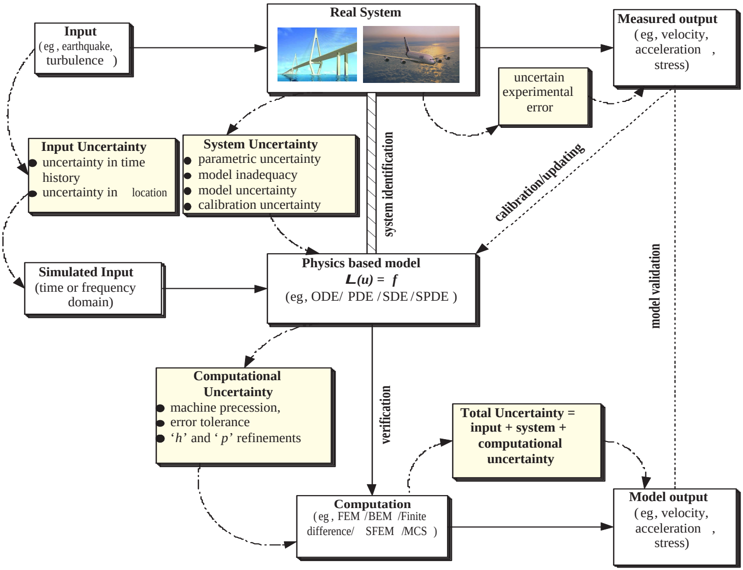
Few general questions

- How does system uncertainty impact the dynamic response? Does it matter?
- What is the underlying physics?
- How can we model uncertainty in dynamic systems? Do we 'know' the uncertainties?
- How can we efficiently quantify uncertainty in the dynamic response for large multi degrees of freedom systems?
- What about using 'black box' type response surface methods?
- Can we use modal analysis for stochastic systems? Does stochastic systems has natural frequencies and mode shapes?

Mathematical models for dynamic systems



A general overview of computational mechanics



Ensembles of structural dynamical systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems). On the other hand, some models are complex!

Complex structural dynamical systems

Complex aerospace system can have millions of degrees of freedom and there can be ‘errors’ and/or ‘lack of knowledge’ in its numerical (Finite Element) model

Model quality

The quality of a model of a dynamic system depends on the following three factors:

- *Fidelity to (experimental) data:*
  The results obtained from a numerical or mathematical model undergoing a given excitation force should be close to the results obtained from the vibration testing of the same structure undergoing the same excitation.
- *Robustness with respect to (random) errors:*
  Errors in estimating the system parameters, boundary conditions and dynamic loads are unavoidable in practice. The output of the model should not be very sensitive to such errors.
- *Predictive capability*
  In general it is not possible to experimentally validate a model over the entire domain of its scope of application. The model should predict the response well beyond its validation domain.

Sources of uncertainty

Different sources of uncertainties in the modeling and simulation of dynamic systems may be attributed, but not limited, to the following factors:

- Mathematical models: equations (linear, non-linear), geometry, damping model (viscous, non-viscous, fractional derivative), boundary conditions/initial conditions, input forces;
- Model parameters: Young's modulus, mass density, Poisson's ratio, damping model parameters (damping coefficient, relaxation modulus, fractional derivative order)
- Numerical algorithms: weak formulations, discretisation of displacement fields (in finite element method), discretisation of stochastic fields (in stochastic finite element method), approximate solution algorithms, truncation and roundoff errors, tolerances in the optimization and iterative methods, artificial intelligent (AI) method (choice of neural networks)
- Measurements: noise, resolution (number of sensors and actuators), experimental hardware, excitation method (nature of shakers and hammers), excitation and measurement point, data processing (amplification, number of data points, FFT), calibration

Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (deterministic)	Unknown	*Analysis (forward problem)*	FEM/BEM/Finite difference
Known (deterministic)	Incorrect (deterministic)	Known (deterministic)	*Updating/calibration*	Modal updating
Known (deterministic)	Unknown	Known (deterministic)	*System identification*	Kalman filter
Assumed (deterministic)	Unknown (deterministic)	Prescribed	*Design*	Design optimisation
Unknown	Partially Known	Known	*Structural Health Monitoring (SHM)*	SHM methods
Known (deterministic)	Known (deterministic)	Prescribed	*Control*	Modal control
Known (random)	Known (deterministic)	Unknown	*Random vibration*	Random vibration methods

Problem-types in structural mechanics

Input	System	Output	Problem name	Main techniques
Known (deterministic)	Known (random)	Unknown	*Stochastic analysis (forward problem)*	SFEM/SEA/RMT
Known (random)	Incorrect (random)	Known (random)	*Probabilistic updating/calibration*	Bayesian calibration
Assumed (random/deterministic)	Unknown (random)	Prescribed (random)	*Probabilistic design*	RBOD
Known (random/deterministic)	Partially known (random)	Partially known (random)	*Joint state and parameter estimation*	Particle Kalman Filter/Ensemble Kalman Filter
Known (random/deterministic)	Known (random)	Known from experiment and model (random)	*Model validation*	Validation methods
Known (random/deterministic)	Known (random)	Known from different computations (random)	*Model verification*	verification methods

- The equations of motion of an undamped non-gyroscopic system with N degrees of freedom can be given by

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{N\times N}$ is the mass matrix, $\mathbf{K} \in \mathbb{R}^{N\times N}$ is the stiffness matrix, $\mathbf{q}(t) \in \mathbb{R}^N$ is the vector of generalized coordinates and $\mathbf{f}(t) \in \mathbb{R}^N$ is the forcing vector.

- Equation (1) represents a set of coupled second-order ordinary-differential equations. The solution of this equation also requires the knowledge of the initial conditions in terms of displacements and velocities of all the coordinates. The *initial conditions* can be specified as

$$\mathbf{q}(0) = \mathbf{q_0} \in \mathbb{R}^N \quad \text{and} \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_\mathbf{0} \in \mathbb{R}^N. \tag{2}$$

- The natural frequencies (ω_j) and the mode shapes ($\mathbf{x}_j$) are intrinsic characteristic of a system and can be obtained by solving the associated matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j, \quad \forall\, j = 1, \cdots, N. \tag{3}$$

- The eigensolutions satisfy the orthogonality condition

$$\mathbf{x}_l^T \mathbf{M}\mathbf{x}_j = \delta_{lj} \tag{4}$$

$$\text{and} \quad \mathbf{x}_l^T \mathbf{K}\mathbf{x}_j = \omega_j^2 \delta_{lj}, \quad \forall\, l, j = 1, \cdots, N \tag{5}$$

- Using the orthogonality relationships in (4) and (5), the equations of motion in the modal coordinates may be obtained as

$$\begin{aligned} \ddot{\mathbf{y}}(t) + \mathbf{\Omega}^2 \mathbf{y}(t) &= \tilde{\mathbf{f}}(t) \\ \text{or} \quad \ddot{y}_j(t) + \omega_j^2 y_j(t) &= \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N \end{aligned} \tag{6}$$

where $\tilde{\mathbf{f}}(t) = \mathbf{X}^T \mathbf{f}(t)$ is the forcing function in modal coordinates.

- Taking the Laplace transform of (1) and considering the initial conditions in (2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \quad (7)$$

$$\text{or} \quad \left[s^2\mathbf{M} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{M}\mathbf{q_0} = \bar{\mathbf{p}}(s) \text{ (say).} \quad (8)$$

- Using the mode orthogonality the response in the frequency domain

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T\bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T\mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + i\omega\mathbf{x}_j^T\mathbf{M}\mathbf{q_0}}{\omega_j^2 - \omega^2}\mathbf{x}_j. \quad (9)$$

This expression shows that the dynamic response of the system is a linear combination of the mode shapes.

Equation of motion

- The equations of motion can expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{10}$$

Theorem

Viscously damped system (10) possesses classical normal modes if and only if $\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$.

- With proportional damping assumption, the damping matrix **C** is simultaneously diagonalizable with **M** and **K**. This implies that the damping matrix in the modal coordinate

$$\mathbf{C}' = \mathbf{X}^T\mathbf{C}\mathbf{X} \tag{11}$$

is a diagonal matrix. The damping ratios ζ_j are defined from the diagonal elements of the modal damping matrix as

$$C'_{jj} = 2\zeta_j\omega_j \quad \forall j = 1, \cdots, N. \tag{12}$$

- The equations of motion in the modal coordinate can be decoupled as

$$\ddot{y}_j(t) + 2\zeta_j\omega_j\dot{y}_j(t) + \omega_j^2 y_j(t) = \tilde{f}_j(t) \quad \forall\, j = 1, \cdots, N. \tag{13}$$

- Taking the Laplace transform of (10) and considering the initial conditions in (2) one has

$$s^2\mathbf{M}\bar{\mathbf{q}} - s\mathbf{M}\mathbf{q_0} - \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + s\mathbf{C}\bar{\mathbf{q}} - \mathbf{C}\mathbf{q_0} + \mathbf{K}\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) \tag{14}$$

$$\text{or} \quad \left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right]\bar{\mathbf{q}} = \bar{\mathbf{f}}(s) + \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{C}\mathbf{q_0} + s\mathbf{M}\mathbf{q_0}. \tag{15}$$

- The transfer function matrix or the receptance matrix can be obtained as

$$\mathbf{H}(i\omega) = \mathbf{X}\left[-\omega^2\mathbf{I} + 2i\omega\boldsymbol{\zeta}\boldsymbol{\Omega} + \boldsymbol{\Omega}^2\right]^{-1}\mathbf{X}^T = \sum_{j=1}^{N} \frac{\mathbf{x}_j\mathbf{x}_j^T}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2}. \tag{16}$$

- The dynamic response in the frequency domain can be conveniently represented as

$$\bar{\mathbf{q}}(i\omega) = \sum_{j=1}^{N} \frac{\mathbf{x}_j^T \bar{\mathbf{f}}(i\omega) + \mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{x}_j^T \mathbf{C}\mathbf{q}_\mathbf{0} + i\omega \mathbf{x}_j^T \mathbf{M}\mathbf{q}_\mathbf{0}}{-\omega^2 + 2i\omega\zeta_j\omega_j + \omega_j^2} \mathbf{x}_j. \tag{17}$$

  Therefore, like undamped systems, the dynamic response of proportionally damped system can also be expressed as a linear combination of the undamped mode shapes.

Dynamic response

- In the time-domain, taking the inverse Laplace transform we have

$$\mathbf{q}(t) = \mathcal{L}^{-1}\left[\bar{\mathbf{q}}(s)\right] = \sum_{j=1}^{N} a_j(t)\mathbf{x}_j \tag{18}$$

where the time dependent constants are given by

$$a_j(t) = \int_0^t \frac{1}{\omega_{d_j}} \mathbf{x}_j^T \mathbf{f}(\tau) e^{-\zeta_j \omega_j (t-\tau)} \sin\left(\omega_{d_j}(t-\tau)\right) d\tau + e^{-\zeta_j \omega_j t} B_j \cos\left(\omega_{d_j} t + \theta_j \tag{19}$$

where

$$B_j = \sqrt{\left(\mathbf{x}_j^T \mathbf{M}\mathbf{q_0}\right)^2 + \frac{1}{\omega_{d_j}^2}\left(\zeta_j \omega_j \mathbf{x}_j^T \mathbf{M}\mathbf{q_0} - \mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} - \mathbf{x}_j^T \mathbf{C}\mathbf{q_0}\right)^2} \tag{20}$$

and $$\tan\theta_j = \frac{1}{\omega_{d_j}}\left(\zeta_j \omega_j - \frac{\mathbf{x}_j^T \mathbf{M}\dot{\mathbf{q}}_\mathbf{0} + \mathbf{x}_j^T \mathbf{C}\mathbf{q_0}}{\mathbf{x}_j^T \mathbf{M}\mathbf{q_0}}\right) \tag{21}$$

Definition of a random variable

- A real random variable $Y(\theta)$, $\theta \in \Theta$ is a set of function defined on Θ such that for every real number y there exist a probability $P(\theta : Y(\omega) \leq y)$
- *Probability Distribution Function:* Consider the event $Y \leq y$. We define

$$F(y) = P(Y \leq y), y \in \mathbb{R}$$

  $F(y)$ is called Probability Distribution Function of Y. $F(y)$ is a monotonically increasing function y with $F(-\infty) = 0$ and $F(\infty) = 1$.
- *Probability Density Function:* The probability structure of a random variable can be described by the derivative of the probability distribution function $p(y)$, called the Probability Density Function. Thus

$$p(y) = \frac{\partial F(y)}{\partial y}$$

  This is normalised such that

$$\int_{-\infty}^{\infty} p(y) \mathrm{d}y = 1$$

- A random field $H(x,\theta)$ is defined as a set function of two arguments $\theta \in \Theta$ and $x \in X$, where Θ is the sample space of the family of random variables $H(x,\bullet)$ and X is the indexing set of parameter X.
- Since a random field $H(x,\theta)$ reduces to a set of random variables at fixed instances of $x = x_1, x_2, \cdots x_n, \cdots$, its probability structure may be defined by a hierarchy of joint probability density function

$$p(h_1, x_1), \quad p(h_1, x_1; h_2, x_2), \cdots, p(h_1, x_1; h_2, x_2; \cdots, h_n, x_n; \cdots) \tag{22}$$

- *Stationary random field:* A random field is said to be stationary if its probability structure is invariant under arbitrary translations of the indexing parameter. Thus $H(x,\theta)$ is stationary if for all $x_1, x_2, \cdots, x_n$ and an arbitrary constant τ if for all n

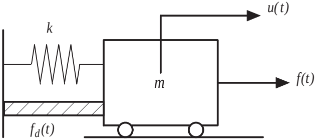
$$p(h_1, x_1; h_2, x_2; \cdots, h_n, x_n) = p(h_1, x_1+\tau; h_2, x_2+\tau; \cdots, h_n, x_n+\tau) \tag{23}$$

- The mean of a random field is given by

$$\mathrm{E}\left[H(x,\theta)\right] = \int H(x,\theta)p(h_1,x_1)\mathrm{d}h_1$$

- The autocorrelation is given by

$$C_{HH}(x_1,x_2) = \int H(x,\theta)p(h_1,x_1;h_2,x_2)\mathrm{d}h_1\mathrm{d}h_2$$



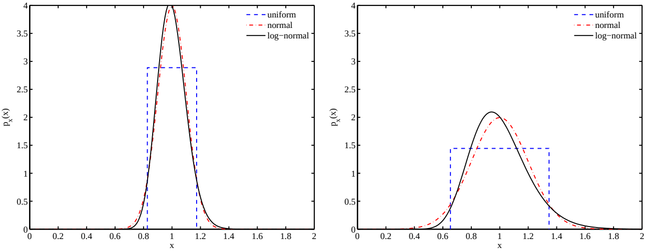
Consider a normalised single degrees of freedom system (SDOF):

$$\ddot{u}(t) + 2\zeta\omega_n\,\dot{u}(t) + \omega_n^2\,u(t) = f(t)/m \qquad (24)$$

Here $\omega_n = \sqrt{k/m}$ is the natural frequency and $\xi = c/2\sqrt{km}$ is the damping ratio.

- We are interested in understanding the motion when the natural frequency of the system is perturbed in a stochastic manner.
- Stochastic perturbation can represent statistical scatter of measured values or a lack of knowledge regarding the natural frequency.

Frequency variability



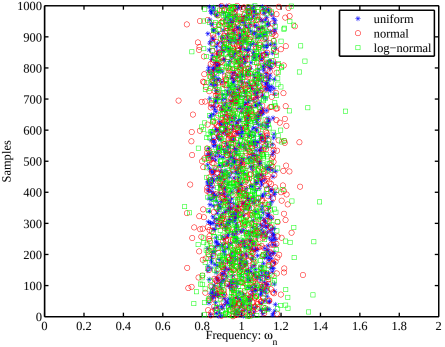
(a) Pdf: $\sigma_a = 0.1$

(b) Pdf: $\sigma_a = 0.2$

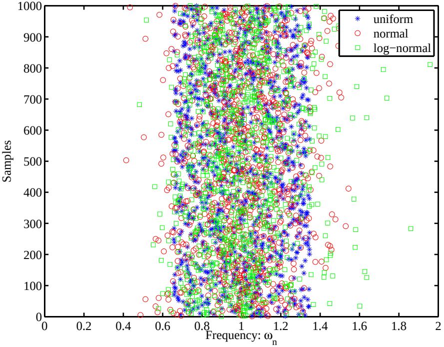
Figure: We assume that the mean of r is 1 and the standard deviation is σ_a.

- Suppose the natural frequency is expressed as $\omega_n^2 = \omega_{n_0}^2 r$, where ω_{n_0} is deterministic frequency and r is a random variable with a given probability distribution function.
- Several probability distribution function can be used.
- We use uniform, normal and lognormal distribution

Frequency samples

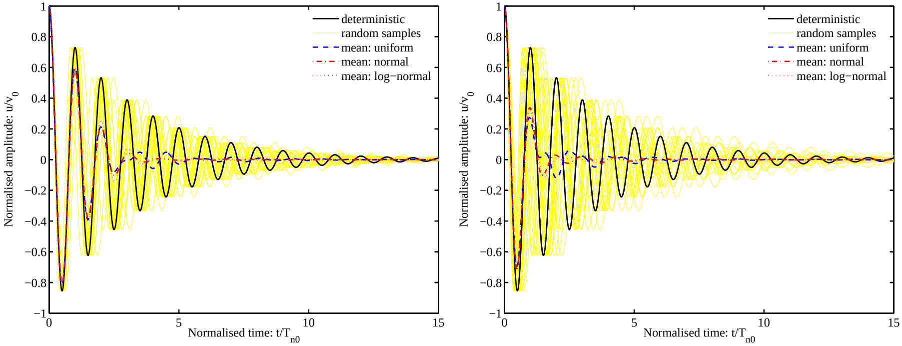


(a) Frequencies: $\sigma_a = 0.1$



(b) Frequencies: $\sigma_a = 0.2$

Figure: 1000 sample realisations of the frequencies for the three distributions

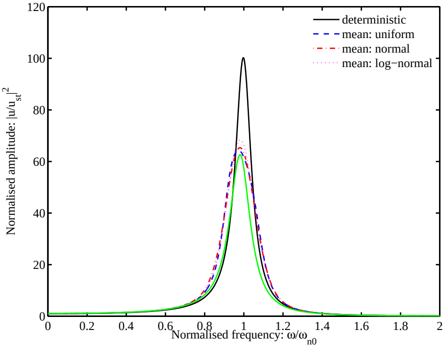


(a) Response: $\sigma_a = 0.1$

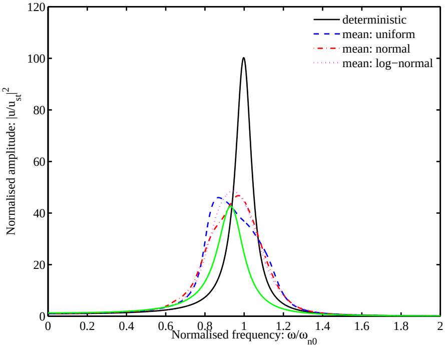
(b) Response: $\sigma_a = 0.2$

Figure: Response due to initial velocity v_0 with 5% damping

Frequency response function



(a) Response: $\sigma_a = 0.1$



(b) Response: $\sigma_a = 0.2$

Figure: Normalised frequency response function $|u/u_{st}|^2$, where $u_{st} = f/k$

- The mean response is more damped compared to deterministic response.
- The higher the randomness, the higher the "effective damping".
- The qualitative features are almost independent of the distribution the random natural frequency.
- We often use averaging to obtain more reliable experimental results - is it always true?

Assuming uniform random variable, we aim to explain some of these observations.

Equivalent damping

- Assume that the random natural frequencies are $\omega_n^2 = \omega_{n_0}^2(1+\epsilon x)$, where x has zero mean and unit standard deviation.
- The normalised harmonic response in the frequency domain

$$\frac{u(\mathrm{i}\omega)}{f/k} = \frac{k/m}{[-\omega^2 + \omega_{n_0}^2(1+\epsilon x)] + 2\mathrm{i}\xi\omega\omega_{n_0}\sqrt{1+\epsilon x}} \qquad (25)$$

- Considering $\omega_{n_0} = \sqrt{k/m}$ and frequency ratio $r = \omega/\omega_{n_0}$ we have

$$\frac{u}{f/k} = \frac{1}{[(1+\epsilon x) - r^2] + 2\mathrm{i}\xi r\sqrt{1+\epsilon x}} \qquad (26)$$

- The squared-amplitude of the normalised dynamic response at $\omega = \omega_{n_0}$ (that is $r = 1$) can be obtained as

$$\hat{U} = \left(\frac{|u|}{f/k}\right)^2 = \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} \tag{27}$$

- Since x is zero mean unit standard deviation uniform random variable, its pdf is given by $p_x(x) = 1/2\sqrt{3}, -\sqrt{3} \le x \le \sqrt{3}$
- The mean is therefore

$$\begin{aligned}
\mathrm{E}\left[\hat{U}\right] &= \int \frac{1}{\epsilon^2 x^2 + 4\xi^2(1+\epsilon x)} p_x(x)\mathrm{d}x \\
&= \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} - \frac{\xi}{\sqrt{1-\xi^2}}\right) \\
&+ \frac{1}{4\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}} \tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}} + \frac{\xi}{\sqrt{1-\xi^2}}\right)
\end{aligned} \tag{28}$$

Equivalent damping

- Note that

$$\frac{1}{2}\left\{\tan^{-1}(a+\delta)+\tan^{-1}(a-\delta)\right\}=\tan^{-1}(a)+O(\delta^2) \tag{29}$$

- Neglecting terms of the order $O(\xi^2)$ we have

$$\mathrm{E}\left[\hat{U}\right]\approx\frac{1}{2\sqrt{3}\epsilon\xi\sqrt{1-\xi^2}}\tan^{-1}\left(\frac{\sqrt{3}\epsilon}{2\xi\sqrt{1-\xi^2}}\right)=\frac{\tan^{-1}(\sqrt{3}\epsilon/2\xi)}{2\sqrt{3}\epsilon\xi} \tag{30}$$

Equivalent damping

- For small damping, the maximum deterministic amplitude at $\omega = \omega_{n_0}$ is $1/4\xi_e^2$ where ξ_e is the equivalent damping for the mean response
- Therefore, the equivalent damping for the mean response is given by

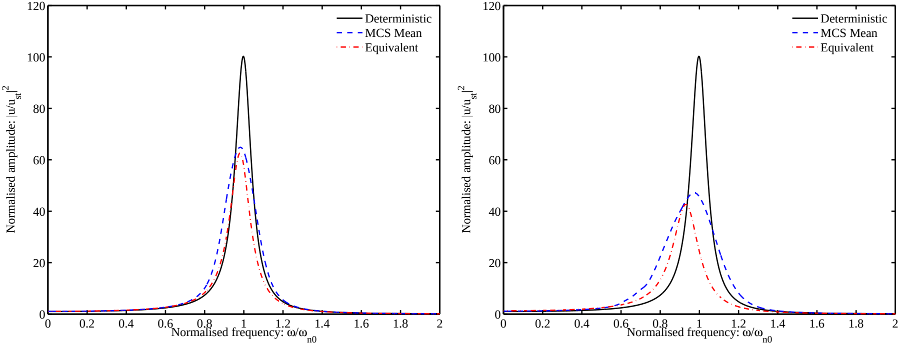
$$(2\xi_e)^2 = \frac{2\sqrt{3}\epsilon\xi}{\tan^{-1}(\sqrt{3}\epsilon/2\xi)} \tag{31}$$

- For small damping, taking the limit we can obtain

$$\xi_e \approx \frac{3^{1/4}\sqrt{\epsilon}}{\sqrt{\pi}}\sqrt{\xi} \tag{32}$$

- *The equivalent damping factor of the mean system is proportional to the square root of the damping factor of the underlying baseline system*

Equivalent frequency response function



(a) Response: $\sigma_a = 0.1$ (b) Response: $\sigma_a = 0.2$

Figure: Normalised frequency response function with equivalent damping ($\xi_e = 0.05$ in the ensembles). For the two cases $\xi_e = 0.0643$ and $\xi_e = 0.0819$ respectively.

Can we extend the ideas based on stochastic SDOF systems to stochastic MDOF systems?

- Stochastic modal analysis to obtain the dynamic response needs further thoughts
- Consider the following 3DOF example:

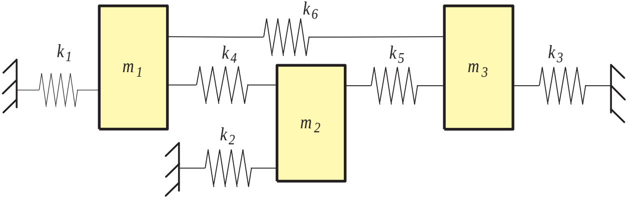
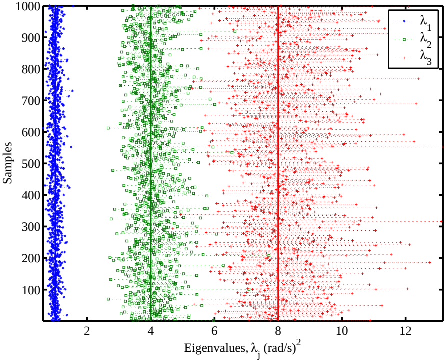
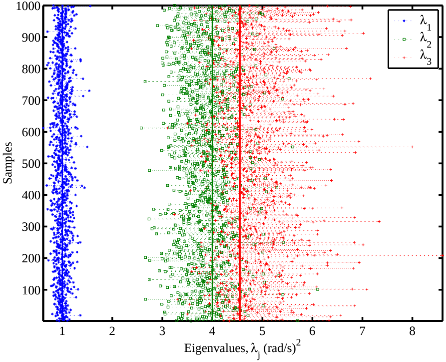


Figure: A 3DOF system with parametric uncertainty in m_i and k_i



(a) Eigenvalues are seperated



(b) Someeigenvalues are close

Figure: Scatter of the eigenvalues due to parametric uncertainties

We consider a stochastic partial differential equation (PDE)

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t,\theta)}{\partial t^2} + \mathcal{L}_\alpha \frac{\partial U(\mathbf{r},t,\theta)}{\partial t} + \mathcal{L}_\beta U(\mathbf{r},t,\theta) = p(\mathbf{r},t) \tag{33}$$

The stochastic operator $\mathcal{L}_\beta$ can be

- $\mathcal{L}_\beta \equiv \frac{\partial}{\partial x} AE(x,\theta) \frac{\partial}{\partial x}$ axial deformation of rods
- $\mathcal{L}_\beta \equiv \frac{\partial^2}{\partial x^2} EI(x,\theta) \frac{\partial^2}{\partial x^2}$ bending deformation of beams

$\mathcal{L}_\alpha$ denotes the stochastic damping, which is mostly proportional in nature. Here $\alpha, \beta : \mathbb{R}^d \times \Theta \to \mathbb{R}$ are stationary square integrable random fields, which can be viewed as a set of random variables indexed by $\mathbf{r} \in \mathbb{R}^d$. Based on the physical problem the random field $a(\mathbf{r},\theta)$ can be used to model different physical quantities (e.g., $AE(x,\theta)$, $EI(x,\theta)$).

- The random process $a(\mathbf{r}, \theta)$ can be expressed in a generalized Fourier type of series known as the Karhunen-Loève expansion

$$a(\mathbf{r}, \theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i}\xi_i(\theta)\varphi_i(\mathbf{r}) \tag{34}$$

- Here $a_0(\mathbf{r})$ is the mean function, $\xi_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1, \mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \nu_j \varphi_j(\mathbf{r}_2), \quad \forall\ j = 1, 2, \cdots \tag{35}$$

Exponential autocorrelation function

The autocorrelation function:

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{36}$$

The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève (KL) expansion in the interval $-a \leq x \leq a$ as

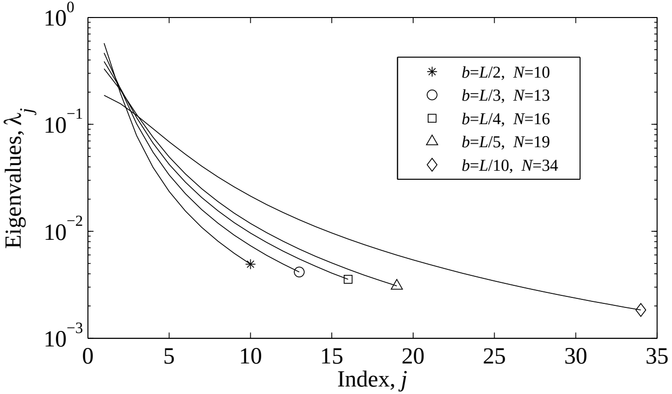
$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \tag{37}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions for odd j and even j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}, \qquad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{c}{\omega_j}, \tag{38}$$

$$\lambda_j = \frac{2c}{{\omega_j}^2 + c^2}, \qquad \varphi_j(x) = \frac{\sin(\omega_j x)}{\sqrt{a - \frac{\sin(2\omega_j a)}{2\omega_j}}}, \qquad \text{where} \quad \tan(\omega_j a) = \frac{\omega_j}{-c}. \tag{39}$$

KL expansion



The eigenvalues of the Karhunen-Loève expansion for different correlation lengths, b, and the number of terms, N, required to capture 90% of the infinite series. An exponential correlation function with unit domain (i.e., $a = 1/2$) is assumed for the numerical calculations. The values of N are obtained such that $\lambda_N/\lambda_1 = 0.1$ for all correlation lengths. Only eigenvalues greater than λ_N are plotted.

Example: A beam with random properties

The equation of motion of an undamped Euler-Bernoulli beam of length L with random bending stiffness and mass distribution:

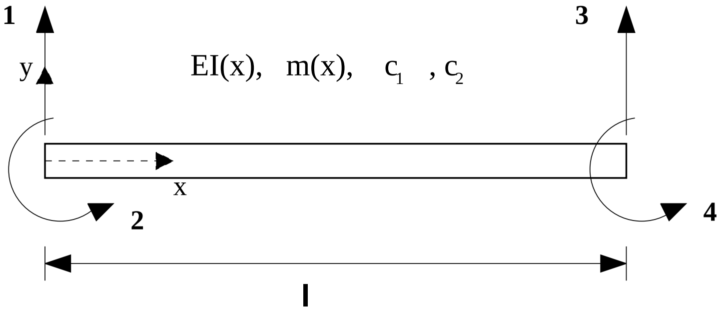
$$\frac{\partial^2}{\partial x^2}\left[EI(x,\theta)\frac{\partial^2 Y(x,t)}{\partial x^2}\right]+\rho A(x,\theta)\frac{\partial^2 Y(x,t)}{\partial t^2}=p(x,t). \tag{40}$$

$Y(x,t)$: transverse flexural displacement, $EI(x)$: flexural rigidity, $\rho A(x)$: mass per unit length, and $p(x,t)$: applied forcing. Consider

$$EI(x,\theta)=EI_0\left(1+\epsilon_1 F_1(x,\theta)\right) \tag{41}$$

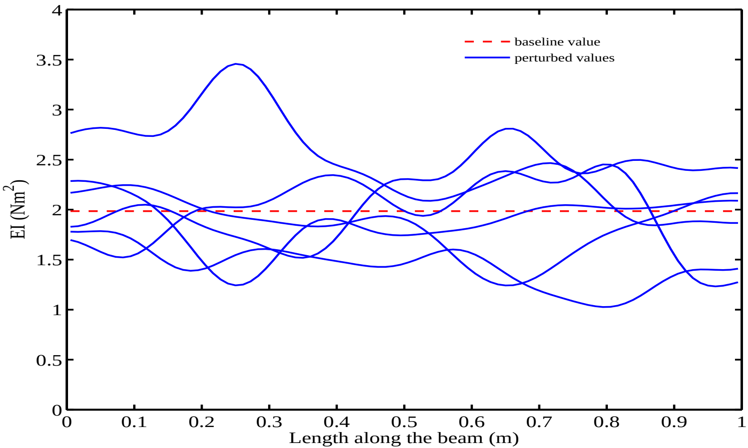
$$\text{and} \quad \rho A(x,\theta)=\rho A_0\left(1+\epsilon_2 F_2(x,\theta)\right) \tag{42}$$

The subscript 0 indicates the mean values, $0<\epsilon_i<<1$ (i=1,2) are deterministic constants and the random fields $F_i(x,\theta)$ are taken to have zero mean, unit standard deviation and covariance $R_{ij}(\xi)$.



Random beam element in the local coordinate.

Realisations of the random field



Some random realizations of the bending rigidity EI of the beam for correlation length $b = L/3$ and strength parameter $\epsilon_1 = 0.2$ (mean 2.0×10^5). Thirteen terms have been used in the KL expansion.

Example: A beam with random properties

We express the shape functions for the finite element analysis of Euler-Bernoulli beams as

$$\mathbf{N}(x) = \mathbf{\Gamma}\, \mathbf{s}(x) \tag{43}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & \frac{-3}{{\ell_e}^2} & \frac{2}{{\ell_e}^3} \\ 0 & 1 & \frac{-2}{{\ell_e}^2} & \frac{1}{{\ell_e}^2} \\ 0 & 0 & \frac{3}{{\ell_e}^2} & \frac{-2}{{\ell_e}^3} \\ 0 & 0 & \frac{-1}{{\ell_e}^2} & \frac{1}{{\ell_e}^2} \end{bmatrix} \quad \text{and} \quad \mathbf{s}(x) = \left[1,\ x,\ x^2,\ x^3\ \right]^T. \tag{44}$$

The element stiffness matrix:

$$\mathbf{K}_e(\theta) = \int_0^{\ell_e} \mathbf{N}''(x) EI(x,\theta) \mathbf{N}''^T(x) dx = \int_0^{\ell_e} EI_0 \left(1 + \epsilon_1 F_1(x,\theta)\right) \mathbf{N}''(x) \mathbf{N}''^T(x) dx. \tag{45}$$

Expanding the random field $F_1(x, \theta)$ in KL expansion

$$\mathbf{K}_e(\theta) = \mathbf{K}_{e0} + \boldsymbol{\Delta}\mathbf{K}_e(\theta) \tag{46}$$

where the deterministic and random parts are

$$\mathbf{K}_{e0} = EI_0 \int_0^{\ell_e} \mathbf{N}^{''}(x)\mathbf{N}^{''^T}(x)\,dx \quad \text{and} \quad \boldsymbol{\Delta}\mathbf{K}_e(\theta) = \epsilon_1 \sum_{j=1}^{N_\mathrm{K}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_{ej}. \tag{47}$$

The constant N_K is the number of terms retained in the Karhunen-Loève expansion and $\xi_{\mathrm{K}j}(\theta)$ are uncorrelated Gaussian random variables with zero mean and unit standard deviation. The constant matrices $\mathbf{K}_{ej}$ can be expressed as

$$\mathbf{K}_{ej} = EI_0 \int_0^{\ell_e} \varphi_{\mathrm{K}j}(x_e + x)\mathbf{N}^{''}(x)\mathbf{N}^{''^T}(x)\,dx \tag{48}$$

The mass matrix can be obtained as

$$\mathbf{M}_e(\theta) = \mathbf{M}_{e_0} + \mathbf{\Delta M}_e(\theta) \tag{49}$$

The deterministic and random parts is given by

$$\mathbf{M}_{e_0} = \rho A_0 \int_0^{\ell_e} \mathbf{N}(x)\mathbf{N}^T(x)\,dx \quad \text{and} \quad \mathbf{\Delta M}_e(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_{ej}. \tag{50}$$

The constant N_{M} is the number of terms retained in Karhunen-Loève expansion and the constant matrices $\mathbf{M}_{ej}$ can be expressed as

$$\mathbf{M}_{ej} = \rho A_0 \int_0^{\ell_e} \varphi_{\mathrm{M}j}(x_e + x)\mathbf{N}(x)\mathbf{N}^T(x)\,dx. \tag{51}$$

Both $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ can be obtained in closed-form.

Example: A beam with random properties

These element matrices can be assembled to form the global random stiffness and mass matrices of the form

$$\mathbf{K}(\theta) = \mathbf{K}_0 + \mathbf{\Delta K}(\theta) \quad \text{and} \quad \mathbf{M}(\theta) = \mathbf{M}_0 + \mathbf{\Delta M}(\theta). \tag{52}$$

Here the deterministic parts $\mathbf{K}_0$ and $\mathbf{M}_0$ are the usual global stiffness and mass matrices obtained form the conventional finite element method. The random parts can be expressed as

$$\mathbf{\Delta K}(\theta) = \epsilon_1 \sum_{j=1}^{N_{\mathrm{K}}} \xi_{\mathrm{K}j}(\theta)\sqrt{\lambda_{\mathrm{K}j}}\mathbf{K}_j \quad \text{and} \quad \mathbf{\Delta M}(\theta) = \epsilon_2 \sum_{j=1}^{N_{\mathrm{M}}} \xi_{\mathrm{M}j}(\theta)\sqrt{\lambda_{\mathrm{M}j}}\mathbf{M}_j \tag{53}$$

The element matrices $\mathbf{K}_{ej}$ and $\mathbf{M}_{ej}$ can be assembled into the global matrices $\mathbf{K}_j$ and $\mathbf{M}_j$. The total number of random variables depend on the number of terms used for the truncation of the infinite series. This in turn depends on the respective correlation lengths of the underlying random fields.

- The equation for motion for stochastic linear MDOF dynamic systems:

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t) + \mathbf{K}(\theta)\mathbf{u}(\theta,t) = \mathbf{f}(t) \tag{54}$$

- $\mathbf{M}(\theta) = \mathbf{M}_0 + \sum_{i=1}^{p} \mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{p} \nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix, $\mathbf{C}(\theta) \in \mathbb{R}^{n\times n}$ as the random damping matrix and $\mathbf{f}(t)$ is the forcing vector
- The mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$). These can be obtained from discretising stochastic fields with a finite number of random variables $(\mu_i(\theta_i)$ and $\nu_i(\theta_i))$ and their corresponding spatial basis functions.
- Proportional damping model is considered for which $\mathbf{C}(\theta) = \zeta_1\mathbf{M}(\theta) + \zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are scalars.

- For the harmonic analysis of the structural system, taking the Fourier transform

$$\left[-\omega^2 \mathbf{M}(\theta) + i\omega \mathbf{C}(\theta) + \mathbf{K}(\theta)\right] \widetilde{\mathbf{u}}(\omega, \theta) = \widetilde{\mathbf{f}}(\omega) \tag{55}$$

  where $\widetilde{\mathbf{u}}(\omega, \theta)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}(\omega)$ is the amplitude of the harmonic force.
- For convenience we group the random variables associated with the mass and stiffness matrices as

$$\xi_i(\theta) = \mu_i(\theta) \quad \text{and} \quad \xi_{j+p_1}(\theta) = \nu_j(\theta) \quad \text{for} \quad i = 1, 2, \ldots, p_1$$
$$\text{and} \quad j = 1, 2, \ldots, p_2$$

- Using $M = p_1 + p_2$ which we have

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i(\omega)\right) \widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}(\omega) \tag{56}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n\times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble.

- For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_0 + \left[i\omega\zeta_2 + 1\right]\mathbf{K}_0, \tag{57}$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{58}$$

$$\text{and} \quad \mathbf{A}_{j+p_1}(\omega) = \left[i\omega\zeta_2 + 1\right]\mathbf{K}_j \quad \text{for} \quad j = 1, 2, \ldots, p_2\,.$$

If the time steps are fixed to Δt, then the equation of motion can be written as

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{C}(\theta)\dot{\mathbf{u}}_{t+\Delta t}(\theta) + \mathbf{K}(\theta)\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t}. \tag{59}$$

Following the Newmark method based on constant average acceleration scheme, the above equations can be represented as

$$\left[a_0\mathbf{M}(\theta) + a_1\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta) \tag{60}$$

$$\text{and,} \quad \mathbf{p}^{eqv}_{t+\Delta t}(\theta) = \mathbf{p}_{t+\Delta t} + f\left(\mathbf{u}_t(\theta), \dot{\mathbf{u}}_t(\theta), \ddot{\mathbf{u}}_t(\theta), \mathbf{M}(\theta), \mathbf{C}(\theta)\right) \tag{61}$$

where $\mathbf{p}^{eqv}_{t+\Delta t}(\theta)$ is the equivalent force at time $t + \Delta t$ which consists of contributions of the system response at the previous time step.

Newmark's method

The expressions for the velocities $\dot{\mathbf{u}}_{t+\Delta t}(\theta)$ and accelerations $\ddot{\mathbf{u}}_{t+\Delta t}(\theta)$ at each time step is a linear combination of the values of the system response at previous time steps (Newmark method) as

$$\ddot{\mathbf{u}}_{t+\Delta t}(\theta) = a_0 \left[\mathbf{u}_{t+\Delta t}(\theta) - \mathbf{u}_t(\theta)\right] - a_2 \dot{\mathbf{u}}_t(\theta) - a_3 \ddot{\mathbf{u}}_t(\theta) \tag{62}$$

$$\text{and,} \quad \dot{\mathbf{u}}_{t+\Delta t}(\theta) = \dot{\mathbf{u}}_t(\theta) + a_6 \ddot{\mathbf{u}}_t(\theta) + a_7 \ddot{\mathbf{u}}_{t+\Delta t}(\theta) \tag{63}$$

where the integration constants $a_i, \ i = 1, 2, \ldots, 7$ are independent of system properties and depends only on the chosen time step and some constants:

$$a_0 = \frac{1}{\alpha \Delta t^2}; \qquad a_1 = \frac{\delta}{\alpha \Delta t}; \qquad a_2 = \frac{1}{\alpha \Delta t}; \qquad a_3 = \frac{1}{2\alpha} - 1; \tag{64}$$

$$a_4 = \frac{\delta}{\alpha} - 1; \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right); \qquad a_6 = \Delta t(1-\delta); \qquad a_7 = \delta \Delta t \tag{65}$$

Following this development, the linear structural system in (60) can be expressed as

$$\underbrace{\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right]}_{\mathbf{A}(\theta)} \mathbf{u}_{t+\Delta t}(\theta) = \mathbf{p}^{eqv}_{t+\Delta t}(\theta). \tag{66}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. For the case of proportional damping, the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written similar to the case of frequency domain as

$$\mathbf{A}_0 = [a_0 + a_1\zeta_1]\,\mathbf{M}_0 + [a_1\zeta_2 + 1]\,\mathbf{K}_0 \tag{67}$$

$$\text{and,} \quad \mathbf{A}_i = [a_0 + a_1\zeta_1]\,\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p_1 \tag{68}$$

$$= [a_1\zeta_2 + 1]\,\mathbf{K}_i \quad \text{for} \quad i = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\,.$$

General mathematical representation

- Whether time-domain or frequency domain methods were used, in general the main equation which need to be solved can be expressed as

$$\left(\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta)\mathbf{A}_i\right)\mathbf{u}(\theta) = \mathbf{f}(\theta) \tag{69}$$

  where $\mathbf{A}_0$ and $\mathbf{A}_i$ represent the deterministic and stochastic parts of the system matrices respectively. These can be real or complex matrices.
- Generic response surface based methods have been used in literature - for example the Polynomial Chaos Method