

# Dynamic Stiffness Method for Stochastic Systems



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# Chapter 1

## Dynamic stiffness method using the weighted integral approach

### 1.1 Model Uncertainty: Why and How a Model Turns Uncertain?

Uncertainties are unavoidable in the description of real-life engineering systems. There are several sources of uncertainties, both in the mathematical models and in the experimental results. Uncertainties can be broadly divided into following three categories. The first type of uncertainty is due to the inherent variability in the system parameters, for example, different cars manufactured from a single production line are not exactly the same. This type of uncertainty is often referred to as *aleatoric uncertainty*. If enough samples are present, it is possible to characterize the variability using well established statistical methods and the probably density functions of the parameters can be obtained. The second type is uncertainty due to lack of knowledge regarding a system. This type of uncertainty is often referred to as *epistemic uncertainty* and generally arise in the modelling of complex systems, for example the problem of predicting cabin noise in helicopters. Due its very nature, it is difficult to quantify or model this type uncertainties. Unlike aleatoric uncertainties, it is recognized that probabilistic models are not quite suitable for epistemic uncertainties. Several possibilistic approaches based on interval algebra, convex sets, Fuzzy sets and generalized Dempster-Schafer theory have been proposed to characterize this type of uncertainties. The third type of uncertainty is similar to the first type except that the corresponding variability characterization is not available, in which case work can be directed to gain better knowledge. This type uncertainty often termed as *prejudicial uncertainty*, may consist of systematic and/or random errors, bias or other prejudices. An example of this type of uncertainty is the use of viscous damping model in spite of knowing that the true damping model is not viscous. The total uncertainty of a system is the combination of these three types of uncertainties. Different sources of uncertainties in the modeling and simulation of dynamic systems may be attributed, but not limited, to the following factors:

- 
- Mathematical models
    - Equations (linear, non-linear)
    - Geometry
    - Damping model (viscous, non-viscous, fractional derivative)
    - Boundary conditions/Initial conditions
    - Input forces
    - Deterministic chaos
  - Model parameters
    - Young's modulus
    - Mass density
    - Poisson's ratio
    - Damping model parameters (damping coefficient, relaxation modulus, fractional derivative order)
  - Numerical algorithms
    - Weak formulations
    - Discretisation of displacement fields (in finite element method)
    - Discretisation of stochastic fields (in stochastic finite element method)
    - Approximate solution algorithms
    - Truncation and roundoff errors
    - Tolerances in the optimization and iterative methods
    - Artificial intelligent (AI) method (choice of neural networks)
  - Surrogate models
    - Choice of model
    - Approximation error
    - Interpolation error
    - Extrapolation error
  - Measurements
    - Noise
    - Resolution (number of sensors and actuators)

- Experimental hardware
- Excitation method (nature of shakers and hammers)
- Excitation and measurement point
- Data processing (amplification, number of data points, FFT)
- Calibration

It is beyond the scope of this note to discuss the quantification and propagation methods of the uncertainties arising from the above sources. We will focus our attention to the modeling and propagation of parametric uncertainties only. Moreover, only probabilistic models of uncertainty will be considered. In what follows next, basic understanding of the probability theory (see Papoulis and Pillai, 2002, for details) will be assumed.

## 1.2 Parametric Uncertainty in Structural Dynamics

The governing equation of motion of a linear structural system with stochastic parameter uncertainties, subjected to external excitations is most often a set of linear differential equation with random coefficients. The problem can be stated as finding the solution of the equation

$$\mathbf{L}(\Omega, \mathbf{r}, t)\mathbf{u}(\Omega, \mathbf{r}, t) = \mathbf{f}(\Omega, \mathbf{r}, t) \quad (1.1)$$

with prescribed boundary conditions and initial conditions. In the above equation  $\mathbf{L}$  is a linear stochastic differential operator,  $\mathbf{u}$  is the random system response to be determined,  $\mathbf{f}$  is the dynamic excitation which can be random,  $\mathbf{r}$  is the special coordinate vector,  $t$  is the time and  $\Omega$  is the sample space denoting the stochastic nature of the problem. Equation (1.1) with  $\mathbf{L}$  as a deterministic operator and  $\mathbf{f}$  as a random forcing function, has been studied extensively within the scope of random vibration theory (Nigam, 1983). Here our interest is when the operator  $\mathbf{L}$  itself is random. There are mainly two methods to model parametric uncertainty using the probabilistic approach: (a) uncertainty modeling using random variables, and (b) uncertainty modeling using stochastic processes. Since we often encounter distributed systems (such as beams, plates shells) during the modeling of real-life systems, stochastic process models should be used for a realistic representation of the uncertainties in the system properties. This in turn will result the operator  $\mathbf{L}$  as a function of stochastic processes. Exact solutions of such stochastic differential equations, even for a simple system such as a standard Euler-Bernoulli beam, are very difficult to obtain. The problem becomes even more intractable when one has to reckon with a real-life engineering dynamics problems, where a set of stochastic boundary value problems defined on irregular spatial domain arises. This has motivated the engineers to seek approximate numerical methods for the solution of governing stochastic differential equations. The methods for solving structural dynamic problems with statistical uncertainties can be broadly grouped under Stochastic

Finite Element Method (SFEM) and Statistical Energy Analysis (SEA). SEA was developed during 1960s (Lyon and Dejong, 1995) to analyze high frequency vibration problems where non-parametric uncertainties plays a key role. The stochastic finite element method is ideally suitable for low-frequency vibration problems where parametric uncertainties plays a key role. Here the stochastic finite element method is explained by applying it to an Euler-Bernoulli beam with stochastic parameter distributions.

### 1.3 Uncertainty Propagation Using Stochastic Finite Element Method

Stochastic finite element method (SFEM) is a generalization of the deterministic finite element method (FEM) to incorporate the random field models for the elastic, mass and damping properties (see the monographs by Ghanem and Spanos, 1991, Kleiber and Hien, 1992). Application of the stochastic finite element method to linear structural dynamics problems typically consists of the following steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions (such as Gaussian/non-Gaussian models).
2. Discretization of random fields, i. e., replacement of the element property random fields by an equivalent finite set of random variables.
3. Formulation of the system equations of motion using stochastic generalization of standard methods such as variational method, energy method, virtual work method or weighted residual method. As a result of this process, the elements of  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  will be random variables.
4. At this point one can take two routes. The first, and the most common approach, is to solve the free vibration problem, which in this case turns out to be a random matrix eigenvalue problem. The aim is to obtain the joint statistics of the mode shapes and natural frequencies. Once they are obtained, the next step is the characterization of response variability for the forced vibration problem.
5. The second route to solve the problem is using the dynamic stiffness method. The main challenge here is to invert the global dynamic stiffness matrix, which in general is a random complex symmetric matrix.

Extensive research works have been done in all of the above mentioned areas during the last few decades. In this section we will discuss the stochastic dynamic stiffness matrix method. The random eigenvalue problems will be discussed in the next chapter.

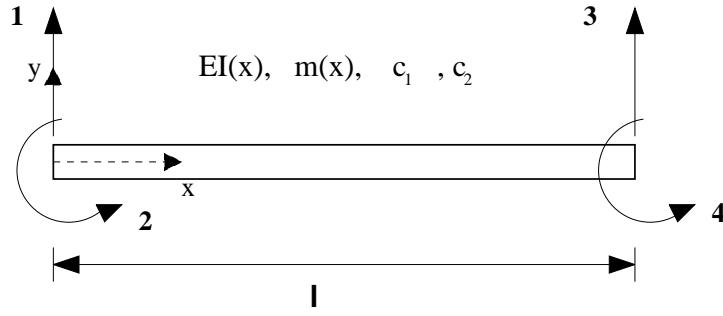


Figure 1.1: Random beam element in the local coordinate.

## 1.4 Stochastic Dynamic Stiffness Matrix Method

We will develop a finite element based formulation to obtain the dynamic stiffness matrix of a general beam element having randomly inhomogeneous mass density, flexural and axial rigidities and elastic foundation modulus. The beam element considered in this study is shown in Figure 1.1. The governing field equation of motion, assuming linear behavior and the validity of the Euler-Bernoulli hypotheses, is given by

$$\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 Y(x, t)}{\partial x^2} + c_1 \frac{\partial^3 Y(x, t)}{\partial x^2 \partial t} \right] + m(x) \frac{\partial^2 Y(x, t)}{\partial t^2} + c_2 \frac{\partial Y(x, t)}{\partial t} = 0 \quad (1.2)$$

Here  $Y(x, t)$  is the transverse flexural displacement,  $EI(x)$  is the flexural rigidity,  $m(x)$  is the mass per unit length,  $c_1$  is the strain rate dependent viscous damping coefficient and  $c_2$  is the velocity dependent viscous damping coefficient. The quantities  $EI(x)$  and  $m(x)$ , in this study, are modeled as meansquare bounded, homogeneous random fields and are taken to have the following form

$$m(x) = m_0 [1 + \epsilon_1 f_1(x)], \quad \text{and} \quad EI(x) = EI_0 [1 + \epsilon_2 f_2(x)]. \quad (1.3)$$

The subscript 0 indicates the mean values,  $\epsilon_1$  and  $\epsilon_2$  are deterministic constants which are usually small compared to the unity. The random fields  $f_1(x)$  and  $f_2(x)$  are taken to have zero mean, unit standard deviation with covariance  $R_{ij}(\xi) = \langle f_i(x) f_j(x) \rangle$  ( $i, j = 1, 2$ ); where  $\langle \bullet \rangle$  denotes the mathematical expectation operator. Since the linear system behavior is being assumed, in steady state, the solution to the field equation can be expressed as

$$Y(x, t) = y(x) \exp [i\omega t]. \quad (1.4)$$

Consequently, the equation governing  $y(x)$  has the form

$$\frac{d^2}{dx^2} \left[ EI(x) \frac{d^2 y}{dx^2} + i\omega c_1 \frac{d^2 y}{dx^2} \right] + [-m(x)\omega^2 + c_2 i\omega] y = 0 \quad (1.5)$$

An exact solution to this type of equation is in general not possible due to the presence of the random fields. The deterministic finite element method will be extended to obtain approximate solutions.

Instead of obtaining the standard mass and stiffness matrices, the problem will be solved using dynamic stiffness method. The dynamic stiffness method is an useful alternative to the more popular mode superposition method of vibration analysis. A vast amount of literature is available on the development of dynamic stiffness method in deterministic context, see, for example Paz (1980), Doyle (1989), Ferguson and Pilkey (1993a,b). Some of the notable features of the method are

- the mass distribution of the element is treated in an exact manner in deriving the element dynamics stiffness matrix;
- the dynamic stiffness matrix of one dimensional structural elements taking into account the effects of flexure, torsion, axial motion, shear deformation effects and damping are exactly determinable, which, in turn, enables the exact vibration analysis of skeletal structures by an inversion of the global dynamic stiffness matrix;
- the method does not employ eigenfunction expansions and, consequently, a major step of the traditional finite element analysis, namely, the determination of natural frequencies and mode shapes, is eliminated which automatically avoids the errors due to series truncation; this makes the method attractive for situations in which a large number of modes participate in vibration;
- the method is essentially a frequency domain approach suitable for steady state harmonic or stationary random excitation problems; generalization to other type of problems through the use of Laplace transforms is also possible;
- the static stiffness matrix and consistent mass matrix appear as the first two terms in the Taylor expansion of the dynamic stiffness matrix in the frequency parameter.

The dynamic stiffness coefficients are, by definition, frequency dependent. When system properties are modeled as random fields, these coefficients become random variables for a fixed value of the driving frequency. As the driving frequency is varied, these stiffness coefficients become random processes evolving in the frequency parameter. Furthermore, the presence of damping in the system makes these random processes complex in nature. The solution procedure is based on the application of finite element method which uses frequency dependent shape functions. This offers a powerful means to discretize the system property random fields and relaxes the dependence of finite element mesh size with respect to the frequency of excitation. The formulation of the dynamic stiffness matrix can be achieved using following steps.

**Step 1** *Derivation of the shape functions:*

Obtain the shape functions using the deterministic undamped field equation. That is, solve the equation

$$\frac{d^4 y}{dx^4} - b^4 y = 0 \quad \text{where} \quad b^4 = \frac{m_0 \omega^2}{EI_0} \quad (1.6)$$



under a set of ‘binary’ boundary conditions. The shape functions  $\{\mathbf{N}(x, \omega)\}$  can be shown to be given by

$$\{\mathbf{N}(x, \omega)\} = [\mathbf{\Gamma}(\omega)]\{\mathbf{s}(x, \omega)\}, \quad (1.7)$$

where

$$[\mathbf{\Gamma}(\omega)] = \begin{bmatrix} \frac{1}{2} \frac{cS+Cs}{cC-1} & -\frac{1}{2} \frac{1+sS-cC}{cC-1} & -\frac{1}{2} \frac{cS+Cs}{cC-1} & \frac{1}{2} \frac{cC+sS-1}{cC-1} \\ \frac{1}{2} \frac{cC+sS-1}{b(cC-1)} & \frac{1}{2} \frac{-Cs+cS}{b(cC-1)} & -\frac{1}{2} \frac{1+sS-cC}{b(cC-1)} & -\frac{1}{2} \frac{-Cs+cS}{b(cC-1)} \\ -\frac{1}{2} \frac{S+s}{cC-1} & \frac{1}{2} \frac{C-c}{cC-1} & \frac{1}{2} \frac{S+s}{cC-1} & -\frac{1}{2} \frac{C-c}{cC-1} \\ \frac{1}{2} \frac{C-c}{b(cC-1)} & -\frac{1}{2} \frac{S-s}{b(cC-1)} & -\frac{1}{2} \frac{C-c}{b(cC-1)} & -\frac{1}{2} \frac{S-s}{b(cC-1)} \end{bmatrix} \quad (1.8)$$

and

$$\{\mathbf{s}(x, \omega)\} = [\sin bx, \cos bx, \sinh bx, \cosh bx]^T \quad (1.9)$$

is the array of basis functions. Here  $C = \cosh bl$ ,  $c = \cos bl$ ,  $S = \sinh bl$  and  $s = \sin bl$ .

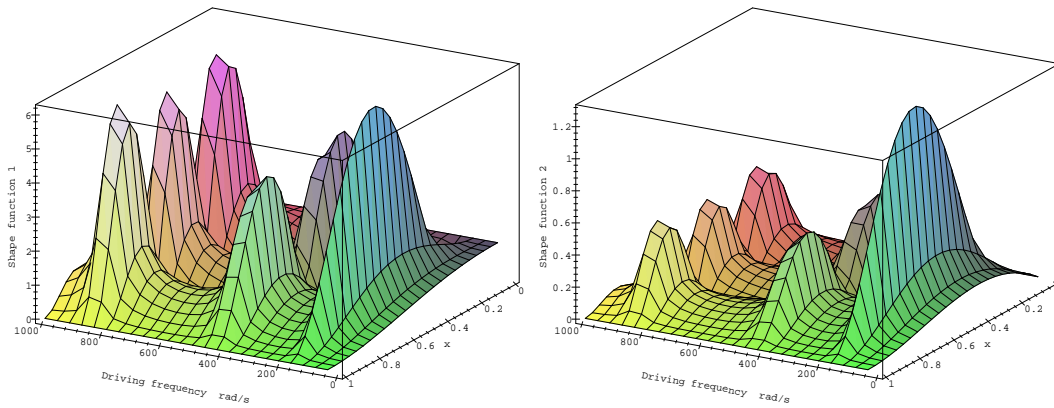


Figure 1.2: Shape functions for element number 2 of the portal frame in

Plots of the first two shape functions, for a typical beam element to be considered later in numerical illustrations, are shown in Figure 1.2. From equation (1.7) It can be derived that, for the static case (that is when  $\omega = 0$ ), the shape functions are cubic polynomials in  $x$  and they agree with the well known beam shape functions. This feature is also observable from Figure 1.2. With increasing value of  $\omega$ , the shape functions adapt themselves and herein lies their major advantage.

### Step 2 Derivation of the Element Equation of Motion:

The displacement field within the element is expressed in the form

$$Y(x, t) = \sum_{j=1}^4 d_j(t) N_j(x, \omega) \quad (1.10)$$

Here  $d_j(t), j = 1, \dots, 4$  are the generalized coordinates representing the nodal displacements. Defining the Lagrangian  $\mathcal{L}(t) = T(t) - V(t)$ , the governing equations for the generalized coordinates  $d_j(t)$  can be obtained from

$$\frac{d}{dt} \left[ \frac{\partial \mathcal{L}}{\partial \dot{d}_j} \right] - \frac{\partial \mathcal{L}}{\partial d_j} = 0; \quad j = 1, \dots, 4. \quad (1.11)$$

**Step 3 Undamped Element Stiffness Matrix:**

Since the motion is harmonic at frequency  $\omega$ , it follows that  $d_j(t) = A_j \exp [i\omega t]$ . Consequently, the undamped dynamic stiffness matrix  $\mathbf{D}_u$  can be shown to be given by

$$\mathbf{D}_u(\omega) = [-\omega^2 I_{ij}(\omega) + J_{ij}(\omega)]_{(4 \times 4)} \quad (1.12)$$

with

$$\begin{aligned} I_{ij}(\omega) &= \int_0^L m(x) N_i(x, \omega) N_j(x, \omega) dx; \\ J_{ij}(\omega) &= \int_0^L \left[ EI(x) \frac{d^2}{dx^2} N_i(x, \omega) \frac{d^2}{dx^2} N_j(x, \omega) \right] dx. \end{aligned} \quad (1.13)$$

Substituting the expression of  $m(x)$  and  $EI(x)$  from equation (1.3) in equations (1.13) and after separating the random and deterministic parts, the undamped element dynamic stiffness matrix can be written as

$$\mathbf{D}_u(\omega) = \bar{\mathbf{D}}_u(\omega) + \sum_{l=1}^{10} [\boldsymbol{\alpha}^l(\omega)] X_l(\omega) \quad (1.14)$$

In the above expression  $\bar{\mathbf{D}}_u(\omega)$  is the deterministic undamped element stiffness matrix which can be shown to be given by

$$\bar{\mathbf{D}}_u(\omega) = \begin{bmatrix} \frac{EI(cS+C_s)b^3}{-1+cC} & \frac{-EI_s b^2 S}{-1+cC} & \frac{EI(S+s)b^3}{-1+cC} & \frac{-EIb^2(C-c)}{-1+cC} \\ \frac{-EI_s b^2 S}{-1+cC} & \frac{EI(-C_s+cS)b}{-1+cC} & \frac{EIb^2(C-c)}{-1+cC} & \frac{-EI(S-s)b}{-1+cC} \\ \frac{EI(S+s)b^3}{-1+cC} & \frac{EIb^2(C-c)}{-1+cC} & \frac{-EI(cS+C_s)b^3}{-1+cC} & \frac{EI_s b^2 S}{-1+cC} \\ \frac{-EIb^2(C-c)}{-1+cC} & \frac{-EI(S-s)b}{-1+cC} & \frac{EI_s b^2 S}{-1+cC} & \frac{EI(-C_s+cS)b}{-1+cC} \end{bmatrix} \quad (1.15)$$

Furthermore,  $X_l, l = 1, \dots, 10$  are random in nature and are given by

$$\begin{aligned} X_1 &= W_{11}; \quad X_2 = W_{12}; \quad X_3 = W_{13}; \quad X_4 = W_{14}; \quad X_5 = W_{22}; \\ X_6 &= W_{23}; \quad X_7 = W_{24}; \quad X_8 = W_{33}; \quad X_9 = W_{34}; \quad X_{10} = W_{44} \end{aligned} \quad (1.16)$$

with

$$W_{kr} = \int_0^L \left[ \{-m_0 \omega^2 \epsilon_1 f_1(x)\} s_k(x) s_r(x) + EI_0 \epsilon_2 f_2(x) \frac{d^2 s_k(x)}{dx^2} \frac{d^2 s_r(x)}{dx^2} \right] dx. \quad (1.17)$$

Also,  $[\boldsymbol{\alpha}^l(\omega)], l = 1, \dots, 10$  are  $4 \times 4$  symmetric matrices of deterministic functions of  $\omega$ . It may be noted that  $X_l(\omega), l = 1, \dots, 10$ , are random processes evolving in the frequency

parameter  $\omega$ . For a fixed value of driving frequency  $\omega$ , these quantities are random variables. The random processes  $X_l(\omega)$  are known as the *dynamic weighted integrals*, since they arise as ‘weighted integrals’ of the random fields  $f_1(x)$  and  $f_2(x)$ . The random variables  $X_l$  are linear functions of the random fields  $f_1(x)$  and  $f_2(x)$  and, therefore, if  $f_1(x)$  and  $f_2(x)$  are modeled as jointly Gaussian random fields, it follows that  $X_l$  are also jointly Gaussian. Furthermore, since the dynamic stiffness coefficients are linear function of  $X_l$ , it follows that these coefficients in turn are also Gaussian distributed. Explicit expressions of the elements of  $[\boldsymbol{\alpha}^l(\omega)]$ ,  $l = 1, \dots, 10$  and  $X_l(\omega)$ ,  $l = 1, \dots, 10$  are given in Appendix B. In addition, MAPLE V based symbolic code, used to generate these expressions are is given in Appendix C .

**Step 4 Damped Element Stiffness Matrix:**

To allow for the effect of damping terms present in the field equation (1.2) the following steps are adopted:

(a) determine the damped dynamic stiffness matrix  $\mathbf{D}_1(\omega)$  for the deterministic homogeneous beam element, that is,  $\mathbf{D}_u(\omega)$  given by equation (1.14) with  $\epsilon_1 = \epsilon_2 = 0$  and

$$b^4 = \frac{m_0\omega^2 - i\omega c_2}{EI_0 + i\omega c_1}; \quad (1.18)$$

(b) determine undamped dynamic stiffness matrix  $\mathbf{D}_2(\omega)$  for the deterministic homogeneous beam element given again by equation (1.14) with  $\epsilon_1 = \epsilon_2 = 0$  and  $b = \frac{m_0\omega^2}{EI_0}$ .

(c) compute the contribution to the dynamic stiffness  $\mathbf{D}_d(\omega)$  from damping terms using  $\mathbf{D}_d(\omega) = \mathbf{D}_1(\omega) - \mathbf{D}_2(\omega)$  and

(d) finally, the damped stochastic dynamic stiffness matrix is obtained as

$$\mathbf{D}(\omega) = \mathbf{D}_u(\omega) + \mathbf{D}_d(\omega) \quad (1.19)$$

Now substituting  $\mathbf{D}(\omega)$  in place of  $\mathbf{D}_u(\omega)$  in equation (12) the complete element dynamic stiffness matrix for an element ‘e’ can be expressed as

$$\mathbf{D}^e(\omega) = \bar{\mathbf{D}}^e(\omega) + \sum_{l=1}^{10} [\boldsymbol{\alpha}^l(\omega)]^e X_l^e \quad (1.20)$$

Here  $\bar{\mathbf{D}}^e(\omega)$  represents the damped deterministic element dynamics stiffness matrix of element ‘e’.

**Step 5 Element Stiffness Matrix in the Global Coordinates:**

The element stiffness matrices in the global coordinates can be written as

$$\mathbf{D}_G^e(\omega) = \mathbf{T}^{eT} \mathbf{D}^e(\omega) \mathbf{T}^e \quad (1.21)$$

where  $\mathbf{T}^e$  denotes the transformation matrix for the element concerned. Now substituting expression of  $\mathbf{D}^e(\omega)$  from equation (1.20) into above equation and separating the deterministic and random parts,  $\mathbf{D}_G^e(\omega)$  can be cast in the form

$$\mathbf{D}_G^e(\omega) = \bar{\mathbf{D}}_G^e(\omega) + \Delta\mathbf{D}_G^e(\omega) \quad (1.22)$$

Here  $\bar{\mathbf{D}}_G^e(\omega) = \mathbf{T}^{eT} \bar{\mathbf{D}}^e(\omega) \mathbf{T}^e$  is the deterministic part of the element stiffness matrix in the global co-ordinates and  $\Delta\mathbf{D}_G^e(\omega)$  is the corresponding random part. In the further analysis, the covariance matrix associated with elements of  $\Delta\mathbf{D}_G^e(\omega)$  would be needed. Also required is the information on the cross covariance between  $\Delta\mathbf{D}_G^e(\omega)$  associated with two distinct elements  $e_u$  and  $e_v$ . This covariance matrix is expressed as follows:

$$\begin{aligned} < \Delta D_{G_{pq}}^{e_u}(\omega) \Delta D_{G_{rs}}^{e_v}(\omega) > = \\ & \sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k=1}^4 \sum_{l=1}^4 \sum_{m=1}^4 \sum_{n=1}^4 T_{kp}^{e_u} T_{lq}^{e_u} T_{mr}^{e_v} T_{ns}^{e_v} [\alpha_{kl}^i]^{e_u} [\alpha_{mn}^j]^{e_v} Cov(X_i^{e_u}, X_j^{e_v}) \end{aligned} \quad (1.23)$$

where  $p, q, r, s = 1, \dots, 4$  and  $e_u, e_v$  runs over the number of element in the structure. The term  $Cov(X_i^{e_u}, X_j^{e_v})$  represents the statistics of the dynamic weighted integrals between two beam elements and is defined in the following section.

#### Step 6 Statistics of the Dynamic Weighted Integrals:

The covariance of the dynamic weighted integrals can be obtained by using equation (1.16) for two different elements of the structure, and can be expressed as

$$Cov(X_i^{e_u}, X_j^{e_v}) = < W_{kr}^{e_u} W_{pq}^{e_v} > = In_{11} + In_{12} + In_{22} \quad (1.24)$$

where

$$In_{11} = m_0^{e_u} m_0^{e_v} \omega^4 \epsilon_1^{e_u} \epsilon_1^{e_v} \int_0^{L_u} \int_0^{L_v} \{s_k^{e_u}(x_1) s_r^{e_u}(x_1) s_p^{e_v}(x_2) s_q^{e_v}(x_2) R_{11}^{e_u e_v}(x_1, x_2)\} dx_1 dx_2 \quad (1.25)$$

$$\begin{aligned} In_{12} = & -\omega^2 \epsilon_1^{e_u} \epsilon_2^{e_v} \int_0^{L_u} \int_0^{L_v} \{m_0^{e_u} EI_0^{e_v} s_k^{e_u}(x_1) s_r^{e_u}(x_1) \frac{d^2 s_p^{e_v}(x_2)}{dx_2^2} \frac{d^2 s_q^{e_v}(x_2)}{dx_2^2} \\ & + m_0^{e_v} EI_0^{e_u} \frac{d^2 s_k^{e_u}(x_1)}{dx_1^2} \frac{d^2 s_r^{e_u}(x_1)}{dx_1^2} s_p^{e_v}(x_2) s_q^{e_v}(x_2)\} R_{12}^{e_u e_v}(x_1, x_2) dx_1 dx_2 \end{aligned} \quad (1.26)$$

$$\begin{aligned} In_{22} = & EI_0^{e_u} EI_0^{e_v} \epsilon_2^{e_u} \epsilon_2^{e_v} \\ & \times \int_0^{L_u} \int_0^{L_v} \left\{ \frac{d^2 s_k^{e_u}(x_1)}{dx_1^2} \frac{d^2 s_r^{e_u}(x_1)}{dx_1^2} \frac{d^2 s_p^{e_v}(x_2)}{dx_2^2} \frac{d^2 s_q^{e_v}(x_2)}{dx_2^2} R_{22}^{e_u e_v}(x_1, x_2) \right\} dx_1 dx_2 \end{aligned} \quad (1.27)$$

In the above equations,  $i, j = 1, \dots, 10$ ; the relationship between  $i, j$  and  $k, r, p, q$  is according to equation (14),  $L_u, L_v$  are lengths of the elements  $e_u$  and  $e_v$  respectively and  $(\bullet)^{e_u}$  represents the properties  $(\bullet)$  corresponding to the element  $e_u$ . The expression  $R_{lm}^{e_u e_v}(x_1, x_2)$ ,  $(l, m = 1, 2)$  appearing in the above equations denotes the covariance function of the random processes  $f_l(x_1)$  and  $f_m(x_2)$  between the elements  $e_u$  and  $e_v$ , that is  $R_{lm}^{e_u e_v}(x_1, x_2) = < f_l^{e_u}(x_1) f_m^{e_v}(x_2) >$ . The random variability in the dynamic stiffness coefficients of the beam element gets characterized completely in terms of the dynamic weighted integrals.

### 1.4.1 Global Dynamic Stiffness Matrix

After having obtained the element dynamic stiffness matrices in the global coordinate system, these matrices can be assembled to derive the the global dynamic stiffness matrix. The rules for assembling the element stiffness matrices are identical to those used in the traditional deterministic finite element analysis. This leads to the expression

$$\mathbf{K}_G(\omega) = \sum_{e=1}^{ne} \mathbf{D}_G^e(\omega) \quad (1.28)$$

where  $\mathbf{K}_G(\omega)$  is the global dynamic stiffness matrix,  $ne$  is the number of elements and  $\mathbf{D}_G^e(\omega)$  is the element dynamic stiffness matrix in the global coordinate system. The summation here implies the addition of appropriate element stiffness matrices at appropriate locations within the global stiffness matrix. The reduced global stiffness matrix  $\mathbf{K}(\omega)$  can be obtained by deleting the rows and columns of  $\mathbf{K}_G(\omega)$  corresponding to the fixed degrees of freedom. The equation of equilibrium is given by

$$\mathbf{K}(\omega)\mathbf{Z}(\omega) = \mathbf{F} \quad (1.29)$$

where  $\mathbf{Z}(\omega)$  is the amplitude of the nodal harmonic displacement vector to be determined and  $\mathbf{F}$  is the amplitude of nodal harmonic force vector and  $\omega$  is the driving frequency. The reduced global dynamic stiffness matrix can be written as

$$\mathbf{K}(\omega) = \mathbf{K}^0(\omega) + \Delta\mathbf{K}(\omega) \quad (1.30)$$

where  $\mathbf{K}^0(\omega)$  is the deterministic part and  $\Delta\mathbf{K}(\omega)$  is the stochastic part. The deterministic part of the matrix is observed to be complex valued and symmetric in nature while the stochastic part is real valued and symmetric. The latter feature arises because (a) the element damping terms are deterministic, and (b) the shape functions are independent of any damping terms. The deterministic part is further represented by

$$\mathbf{K}^0(\omega) = [\mathbf{K}_R^0(\omega) + i\mathbf{K}_I^0(\omega)] \quad (1.31)$$

where  $\mathbf{K}_R^0(\omega)$  and  $\mathbf{K}_I^0(\omega)$  are respectively, the real and imaginary parts of  $\mathbf{K}^0(\omega)$ . When element property random fields arise as Gaussian fields, the elements of stochastic part of the dynamic element stiffness matrix also become Gaussian distributed. Furthermore, since the global dynamic stiffness matrix is obtained by linear superpositioning of element stiffness matrices, it follows that elements of  $\Delta\mathbf{K}(\omega)$  are also Gaussian distributed. Thus, the complete description of  $\mathbf{K}(\omega)$  is given by its mean  $\mathbf{K}^0(\omega)$  and the covariance matrix of the weighted integrals associated with the  $ne$  number of finite elements.

### 1.4.2 Inversion of the Global Dynamic Stiffness Matrix

The problem of determination of  $\mathbf{Z}(\omega)$  requires the inversion of the global dynamic stiffness matrix. Although for a fixed value of  $\omega$  the elements of  $\mathbf{K}(\omega)$  are complex valued *Gaussian*

random variables, upon inversion, the elements of  $\mathbf{K}^{-1}(\omega)$  and consequently  $\mathbf{Z}(\omega)$  will in general be non-Gaussian. Since  $\mathbf{Z}(\omega)$  is complex valued, it can be described either in terms of its real and imaginary parts, or alternatively, by its amplitude and phase vectors. In the latter case a further nonlinear transformation of real and imaginary parts of  $\mathbf{Z}(\omega)$  is implied. Three alternative strategies to find the inverse of  $\mathbf{K}(\omega)$  will be presented. The first two approaches are analytical in nature while the last employs simulation techniques.

#### *Random Eigenfunction Expansion Method*

The equation of equilibrium (1.29) represents a set of linear random algebraic equations in  $\mathbf{Z}(\omega)$ . We seek an approximate solution to this equation of the form

$$\hat{\mathbf{Z}} = [\Phi]\{\mathbf{a}\} \quad (1.32)$$

where  $\Phi$  represents a set of random basis vectors with known joint statistics and  $\mathbf{a}$  is a vector of unknown complex valued deterministic constants. These unknown constants are determined by adopting a Galerkin type of error minimization scheme which leads to the expressions for the unknown  $\mathbf{a}$  in terms of the statistics of elements of  $\Phi$ . In principle any set of basis functions can be chosen to represent the unknown  $\mathbf{Z}(\omega)$ : in this study, these functions are taken to be the eigenvectors of the real part of the reduced global dynamic stiffness matrix. That is, the basis vectors are taken to be the eigenvectors of the matrix  $\mathbf{K}_R(\omega) = \mathbf{K}_R^0(\omega) + \Delta\mathbf{K}(\omega)$ . It may be noted that this matrix is symmetric, real valued, positive definite and its elements form a set of Gaussian random variables. The statistics of the eigenvectors of this matrix can be determined by solving the random eigenvalue problem given by

$$[\mathbf{K}_R^0(\omega) + \Delta\mathbf{K}(\omega)] \phi = \lambda\phi \quad (1.33)$$

This type of random eigenvalue problems typically arise in the determination of natural frequencies and buckling loads of randomly parametered structures and has attracted the attention of many researches in the past. We borrow these results and adopt them to derive a suitable set of basis eigenvectors. The random eigensolutions that are being used in this study do not have any direct physical meaning. The details of the perturbation method used to solve the random eigenvalue problem will be discussed later.

To determine the unknown constant  $\mathbf{a}$ , we begin by substituting  $\hat{\mathbf{Z}}$  in equation (1.29) and define the error vector  $\{\xi\}$  as

$$\{\xi\}_{(n \times 1)} = \mathbf{K}(\omega)[\Phi]\{\mathbf{a}\} - \mathbf{F} = [\mathbf{K}^r(\omega) + i\mathbf{K}_I^0(\omega)] [\Phi]\{\mathbf{a}\} - \mathbf{F} \quad (1.34)$$

Adopting the Galerkin weighted residual method, we impose the conditions

$$\langle \phi_j, \xi \rangle = 0; \quad \text{for } j = 1, \dots, n \quad (1.35)$$

where  $\phi_j$  is the  $j$ th column of the matrix  $\Phi$  which plays the role of the weighting function. This equation can be recast in a matrix form as

$$\langle [\Phi]^T \{\xi\} \rangle = 0 \quad (1.36)$$

Substituting  $\{\boldsymbol{\xi}\}$  from equation (1.34) into the above equation we obtain

$$[\langle \boldsymbol{\Phi}^T \mathbf{K}^r(\omega) \boldsymbol{\Phi} \rangle + i \langle \boldsymbol{\Phi}^T \mathbf{K}_I^0(\omega) \boldsymbol{\Phi} \rangle] \{\mathbf{a}\} = \langle \boldsymbol{\Phi}^T \mathbf{F} \rangle$$

which leads to

$$[[\bar{\boldsymbol{\lambda}}] + i \langle \boldsymbol{\Phi}^T \mathbf{K}_I^0(\omega) \boldsymbol{\Phi} \rangle] \{\mathbf{a}\} = \langle \boldsymbol{\Phi}^T \rangle \mathbf{F} \quad (1.37)$$

The above equation can further be written concisely as

$$\mathbf{G}\{\mathbf{a}\} = \mathbf{P} \quad (1.38)$$

where

$$\mathbf{G} = [\bar{\boldsymbol{\lambda}}] + i \langle \boldsymbol{\Phi}^T \mathbf{K}_I^0(\omega) \boldsymbol{\Phi} \rangle \quad \text{and} \quad \mathbf{P} = \langle \boldsymbol{\Phi}^T \rangle \mathbf{F}. \quad (1.39)$$

Here  $[\bar{\boldsymbol{\lambda}}]$  is a diagonal matrix containing the mean eigenvalues. The matrix  $\mathbf{G}$  in the above equation can be written in the index form as

$$G_{ij} = \delta_{ij} \bar{\lambda}_i + i \bar{w}_{ij}$$

where  $\delta_{ij}$  is the Kronecker delta function and  $\bar{w}_{ij}$  is given by

$$\bar{w}_{ij} = \langle w_{ij} \rangle = \sum_{r=1}^n \sum_{s=1}^n \langle \phi_{ri} \phi_{sj} \rangle K^0_{Irs} \quad (1.40)$$

The term  $\langle \phi_{ri} \phi_{sj} \rangle$  in the above equation represents the correlation between  $r$ th element of  $i$ th eigenvector and  $s$ th element of  $j$ th eigenvector. This term is given by

$$\begin{aligned} \langle \phi_{ri} \phi_{sj} \rangle &= \sum_{l=1}^n \sum_{m=1}^n C_{rl} C_{sm} \\ &\times \left[ (1 - \delta_{li})(1 - \delta_{mj}) \frac{1}{\mu_{li} \mu_{mj}} \sum_{p=1}^n \sum_{q=1}^n \sum_{t=1}^n \sum_{k=1}^n C_{pl} C_{qi} C_{tm} C_{kj} \langle \Delta K_{pq}(\omega) \Delta K_{tk}(\omega) \rangle \right] \end{aligned} \quad (1.41)$$

Here  $\mu_{ij} = \mu_i - \mu_j$  and  $\mu_i$  denotes the eigenvalues of the real deterministic matrix  $\mathbf{K}_R^0(\omega)$ ,  $C_{ij}$  are the component of  $\mathbf{C}$  which is the matrix containing the unity normalized eigenvectors of  $\mathbf{K}_R^0(\omega)$ . The quantity  $\langle \Delta K_{pq}(\omega) \Delta K_{tk}(\omega) \rangle$  denotes the correlations between the stiffness coefficients which in turn are expressible in terms of the statistics of the weighted integrals. Thus, having determined the matrix  $\mathbf{G}$ , the unknown constants  $\mathbf{a}$  can now be determined using equation (1.38) which further leads to

$$\mathbf{Z}(\omega) = [\boldsymbol{\Phi}] \{\mathbf{a}\} \quad (1.42)$$

Using the first-order perturbation it can be shown that the elements of  $[\boldsymbol{\Phi}]$ , that is,  $\phi_j, j = 1, \dots, n$ , are Gaussian distributed. From equation (1.42) it follows that the solution vector  $\mathbf{Z}(\omega)$ , for a fixed value of  $\omega$ , is also a vector of Gaussian random variables. This feature facilitates the evaluation of the statistics of the amplitude and phase vectors associate with

$\mathbf{Z}(\omega)$ . To achieve this, we separate  $\mathbf{Z}(\omega)$  and  $\mathbf{a}$  into their respective real and imaginary parts and write

$$z_i = z_i^R + iz_i^I \quad \text{where} \quad z_i^R = \sum_{j=1}^n \phi_{ij} a_j^R, \quad \text{and} \quad z_i^I = \sum_{j=1}^n \phi_{ij} a_j^I \quad (1.43)$$

with

$$\begin{aligned} \langle z_i^R \rangle &= \sum_{j=1}^n C_{ij} a_j^R; & \langle z_i^I \rangle &= \sum_{j=1}^n C_{ij} a_j^I \\ \langle z_i^{R^2} \rangle &= \sum_{j=1}^n \sum_{k=1}^n \langle \phi_{ij} \phi_{ik} \rangle a_j^R a_k^R; & \langle z_i^{I^2} \rangle &= \sum_{j=1}^n \sum_{k=1}^n \langle \phi_{ij} \phi_{ik} \rangle a_j^I a_k^I \\ \langle z_i^R z_i^I \rangle &= \sum_{j=1}^n \sum_{k=1}^n \langle \phi_{ij} \phi_{ik} \rangle a_j^R a_k^I \end{aligned} \quad (1.44)$$

In the above expressions  $(\bullet)^R$  and  $(\bullet)^I$  are respectively the real and imaginary part of  $(\bullet)$ . The second order statistics of  $z_i^R$  and  $z_i^I$  are dependent upon joint statistics of the basis random vectors. Subsequently the moments of amplitude and phase of the elements of  $\mathbf{Z}(\omega)$  are given by

$$\begin{aligned} \langle |z_i| \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{z_i^{R^2} + z_i^{I^2}} p_{z_i^R z_i^I}(z_i^R, z_i^I) dz_i^R dz_i^I \\ \langle |z_i|^2 \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (z_i^{R^2} + z_i^{I^2}) p_{z_i^R z_i^I}(z_i^R, z_i^I) dz_i^R dz_i^I \end{aligned} \quad (1.45)$$

$$\begin{aligned} \langle \arg(z_i) \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \tan^{-1} \left( \frac{z_i^I}{z_i^R} \right) \right\} p_{z_i^R z_i^I}(z_i^R, z_i^I) dz_i^R dz_i^I \\ \langle (\arg(z_i))^2 \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \tan^{-1} \left( \frac{z_i^I}{z_i^R} \right) \right\}^2 p_{z_i^R z_i^I}(z_i^R, z_i^I) dz_i^R dz_i^I \end{aligned} \quad (1.46)$$

In the above equations  $p_{z_i^R z_i^I}(z_i^R, z_i^I)$  is the two dimensional joint probability density function of  $z_i^R$  and  $z_i^I$ . This is completely characterized by the mean, standard deviation and correlation coefficient of  $z_i^R$  and  $z_i^I$  which can be obtained from equations (1.44).

#### Complex Neumann Expansion Method

Shinozuka and Yamazaki (1988) have applied the Neumann expansion method to the inversion of static stiffness matrix in stochastic finite element applications. We begin by writing the equilibrium equation in the following form:

$$[\mathbf{K}_R^0(\omega) + i\mathbf{K}_I^0(\omega) + \Delta\mathbf{K}(\omega)] \mathbf{Z}(\omega) = \mathbf{F} \quad (1.47)$$

Let  $\mathbf{Z}^0(\omega)$  denote the solution in absence of system randomness. This is given by the solution of the equation

$$\mathbf{K}^0(\omega) \mathbf{Z}^0(\omega) = \mathbf{F} \quad (1.48)$$



where  $\mathbf{K}^0(\omega) = [\mathbf{K}_R^0(\omega) + i\mathbf{K}_I^0(\omega)]$  is the deterministic damped dynamic stiffness matrix. According to the Neumann expansion

$$\mathbf{Z}(\omega) = [\mathbf{K}^0(\omega) + \Delta\mathbf{K}(\omega)]^{-1}\mathbf{F} = [\mathbf{I} - \mathbf{R} + \mathbf{R}^2 - \mathbf{R}^3 + \dots] \mathbf{Z}^0(\omega) \quad (1.49)$$

with

$$\mathbf{R} = \mathbf{K}^{0-1}(\omega)\Delta\mathbf{K}(\omega) = [\mathbf{K}_R^0(\omega) + i\mathbf{K}_I^0(\omega)]^{-1}\Delta\mathbf{K}(\omega) \quad (1.50)$$

and  $\mathbf{I}$  is the unit matrix. Here  $\mathbf{R}$  is complex valued and is random in nature. Since the elements of  $\Delta\mathbf{K}(\omega)$  are Gaussian distributed, it follows that elements of real and imaginary parts of  $\mathbf{R}$  are also Gaussian distributed. If only the first-order terms are retained in the series expansion (1.49), the response vector  $\mathbf{Z}(\omega)$  will have Gaussian distributed elements. Thus, retaining only the linear terms in  $\mathbf{R}$ , the response vector  $\mathbf{Z}(\omega)$  is given by

$$\mathbf{Z}(\omega) = [\mathbf{I} - \mathbf{R}] \mathbf{Z}^0(\omega) \quad \text{or} \quad z_i = z_i^o - \sum_{j=1}^n R_{ij} z_j^o \quad (1.51)$$

For notational convenience writing  $\mathbf{Q} = \mathbf{K}_0^{-1}(\omega)$  equation (1.50) can be rewritten as

$$\mathbf{R} = \mathbf{Q}\Delta\mathbf{K}(\omega) \quad \text{or} \quad R_{ij} = \sum_{l=1}^n Q_{il}\Delta K_{lj} \quad (1.52)$$

Substituting  $R_{ij}$  from equation (1.52) into equation (1.51), the elements of the solution vector can be obtained as

$$z_i = z_i^o - \sum_{j=1}^n \sum_{l=1}^n Q_{il} z_j \Delta K_{lj} \quad (1.53)$$

These elements are complex quantities with real and imaginary parts being Gaussian distributed. Denoting by  $z_i^R$  and  $z_i^I$  the real and imaginary parts of  $z_i$  respectively, we get

$$z_i^R = z_i^{0R} - \sum_{j_1=1}^n \sum_{l_1=1}^n \left( Q_{il_1}^R z_{j_1}^{0R} - Q_{il_1}^I z_{j_1}^{0I} \right) \Delta K_{l_1 j_1} \quad (1.54)$$

$$z_i^I = z_i^{0I} - \sum_{j_2=1}^n \sum_{l_2=1}^n \left( Q_{il_2}^I z_{j_2}^{0R} + Q_{il_2}^R z_{j_2}^{0I} \right) \Delta K_{l_2 j_2} \quad (1.55)$$

These expressions are further simplified to get

$$z_i^R = z_i^{0R} - U_i^R; \quad z_i^I = z_i^{0I} - U_i^I \quad (1.56)$$

with

$$\langle z_i^R \rangle = z_i^{0R}, \quad \langle z_i^I \rangle = z_i^{0I}, \quad (1.57)$$

$$\text{Var}(z_i^R) = \langle U_i^R{}^2 \rangle =$$

$$\sum_{j_1=1}^n \sum_{l_1=1}^n \sum_{j_2=1}^n \sum_{l_2=1}^n \left( Q^R_{il_1} z^{0R}_{j_1} - Q^I_{il_1} z^{0I}_{j_1} \right) \left( Q^R_{il_2} z^{0R}_{j_2} - Q^I_{il_2} z^{0I}_{j_2} \right) < \Delta K_{l_1 j_1} \Delta K_{l_2 j_2} >, \quad (1.58)$$

$$\begin{aligned} \text{Var}(z^I_i) &= \langle U^I_i{}^2 \rangle = \\ & \sum_{j_1=1}^n \sum_{l_1=1}^n \sum_{j_2=1}^n \sum_{l_2=1}^n \left( Q^I_{il_1} z^{0R}_{j_1} + Q^R_{il_1} z^{0I}_{j_1} \right) \left( Q^I_{il_2} z^{0R}_{j_2} + Q^R_{il_2} z^{0I}_{j_2} \right) < \Delta K_{l_1 j_1} \Delta K_{l_2 j_2} >, \end{aligned} \quad (1.59)$$

and

$$\begin{aligned} \text{Cov}(z^R_i z^I_i) &= \langle U^R_i U^I_i \rangle = \\ & \sum_{j_1=1}^n \sum_{l_1=1}^n \sum_{j_2=1}^n \sum_{l_2=1}^n \left( Q^R_{il_1} z^{0R}_{j_1} - Q^I_{il_1} z^{0I}_{j_1} \right) \left( Q^I_{il_2} z^{0R}_{j_2} + Q^R_{il_2} z^{0I}_{j_2} \right) < \Delta K_{l_1 j_1} \Delta K_{l_2 j_2} >. \end{aligned} \quad (1.60)$$

As has been noted already,  $z^R_i$  and  $z^I_i$ , for a fixed value of  $\omega$ , are Gaussian distributed random variables. Thus, using the above expressions, the joint probability density function of  $z^R_i$  and  $z^I_i$  can be derived. Consequently, the moments of amplitude and phase of  $z_i$  can be evaluated using equations (1.45,1.46).

#### *Combined Analytical and Simulation Method*

The analytical methods presented in the preceding two sections introduce approximations at the stage of inverting the random dynamic stiffness matrix. These approximations are in addition to those involved in discretizing the random fields to formulate the element stiffness matrices and in handling damping properties. The approximations associated with inverting the matrix can be avoided if one adopts Monte Carlo simulation strategy to invert the reduced global stiffness matrix. This would require

- A. digital simulation of samples of  $\Delta \mathbf{K}(\omega)$ ; as has been already noted, for a fixed  $\omega$  the elements of  $\Delta \mathbf{K}(\omega)$  are a set of Gaussian distributed random variables and these can be easily simulated;
- B. simulation of samples of  $\mathbf{Z}(\omega)$  by numerically inverting  $\mathbf{K}(\omega)$ , and
- C. statistical processing of samples of  $\mathbf{Z}(\omega)$  to arrive at the required statistics of amplitude and phase of  $\mathbf{Z}(\omega)$ .

This method treats the inversion of the stiffness matrix in an exact manner while the other steps in the solution continue to be approximate in nature. The source of approximation

here is associated with the discretization of the random fields, treatment of damping and also with the use of limited number of samples for estimating the response statistics. An advantage of this method over the other two methods presented so far is that it leads to non-Gaussian estimates for elements of  $\mathbf{Z}(\omega)$ .

### 1.4.3 Full Scale Digital Simulations

Given the approximate nature of the methods presented in the previous section, it is essential that their performance be verified by comparing the results obtained using these methods with those from more exact procedures. Such procedures can be formulated by combining the Monte Carlo simulation procedures with traditional normal mode expansion based finite element procedures, transfer matrix techniques or direct dynamic stiffness matrix approach. In this study we resort to the last mentioned approach. This consists of the following steps:

1. discretize the given structure into as many elements as is the number of beams: thus, a three leg portal frame gets discretized into three elements;
2. digital simulation of samples of  $EI(x)$  and  $m(x)$  for all the elements considered;
3. derivation of sample dynamic stiffness matrix for each of the elements; this requires the harmonic response analysis of inhomogeneous beam elements; the procedure described by Sarkar and Manohar (1996), Manohar and Adhikari (1998a) has been used for this purpose; this is based on converting the governing boundary value problems into a set of equivalent initial value problems and integrating these resulting equations numerically using Runge-Kutta algorithm; this leads to solutions which are ‘exact’ within the framework of accuracy of Runge-Kutta algorithms;
4. formulation of sample global stiffness matrix and its numerical inversion leading to sample solution vectors;
5. statistical processing of the ensemble of response vectors to obtain the desire statistics

In this approach, the three major steps of the response analysis, namely, the discretization of random fields, the treatment of damping terms and the inversion of random stiffness matrix, are all handled in an exact manner with the only source of approximation being the finite size of samples used to estimate the response statistics. In terms of computational effort needed, this method obviously is more demanding than the three methods discussed in the previous section.

### 1.4.4 Numerical Results And Discussions

To illustrate the relative performance of the different formulations, the harmonic response analysis of a three leg portal frame shown in Figure 1.3 is considered. The flexural rigidity

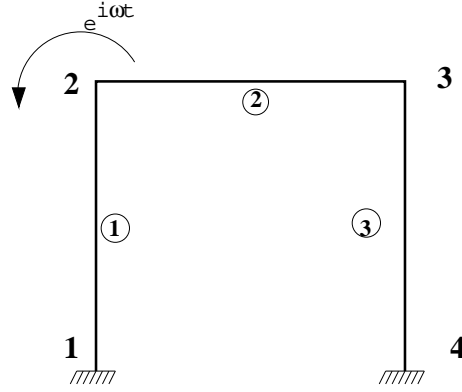


Figure 1.3: Finite element model for dynamic stiffness analysis of portal frame;  $EI_0=10.0$ ,  $L=1.0$ ,  $m_0=0.2$ ,  $\epsilon_1=0.05$ ,  $\epsilon_2=0.05$ ,  $c_1 = 0$ ,  $c_2=1.0$  (same for all the three elements).

and mass density for each element are taken to be independent, homogeneous, Gaussian random fields. It is assumed that  $\epsilon_1 = \epsilon_2 = 0.05$ , and the autocovariance of the processes  $f_1(x)$  and  $f_2(x)$  for all the three beam elements are taken to be of the form

$$R_{ii}(\xi) = \cos\lambda_i\xi; \quad i = 1, 2 \quad (1.61)$$

with  $\lambda_i = 10\pi$  per unit length. The particular choice of autocovariance function and its parameters in this example is made only for purpose of illustrations and the theoretical results developed are expected to be valid for other forms of these functions also. It is also assumed that the random properties of distinct elements are mutually uncorrelated. It may be recalled that the harmonic displacement amplitudes for damped structural systems are complex valued. When the structural element is random, the displacement amplitudes can be interpreted as complex valued random processes evolving in the frequency parameter  $\omega$ . Let us focus our attention on evaluating the mean and standard deviation of the amplitude and phase of the side sway as a function of the driving frequency  $\omega$ . The results of theoretical analysis are compared with a 500 samples full scale Monte Carlo simulation in figures 1.4-1.8. The algorithm used in the simulation work to simulate samples of the dynamic element stiffness matrix is as outlined by Sarkar and Manohar (1996), Manohar and Adhikari (1998a). The simulation work does not involve discretization of the random fields, it treats the damping terms appearing in equation (1.5) exactly and inverts the random complex matrix in an exact way. Therefore, the simulation results serve to evaluate several aspects of the approximate analytical procedures.

The theoretical predictions generally compare well with the simulations results over the entire frequency range considered. This supports the approximations made in the treatment of system randomness and damping in deriving the element dynamic stiffness matrix and also in inverting the random global dynamic stiffness matrix to calculate the displacement amplitudes. At resonance points, the theory and simulations compare better than at the anti-resonance points. The numerical results on response statistics indicate that the ampli-

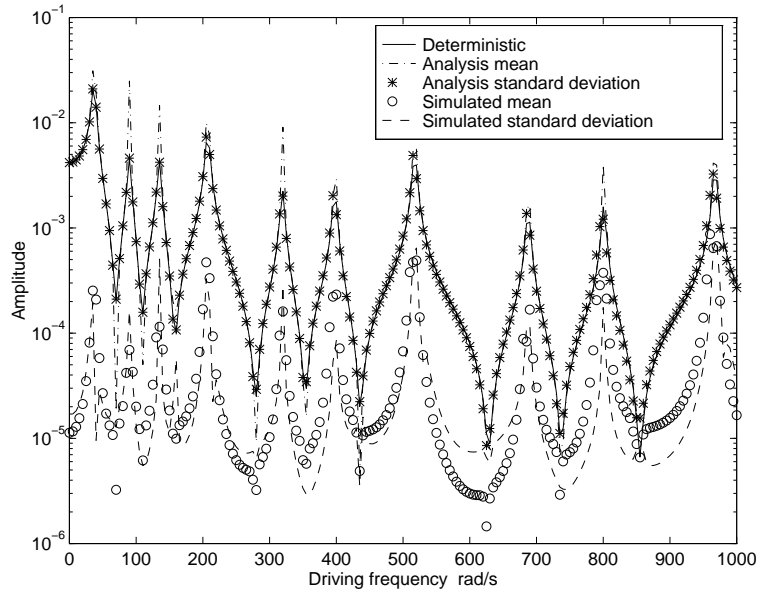


Figure 1.4: Statistics of amplitude of side sway using Eigenfunction expansion method

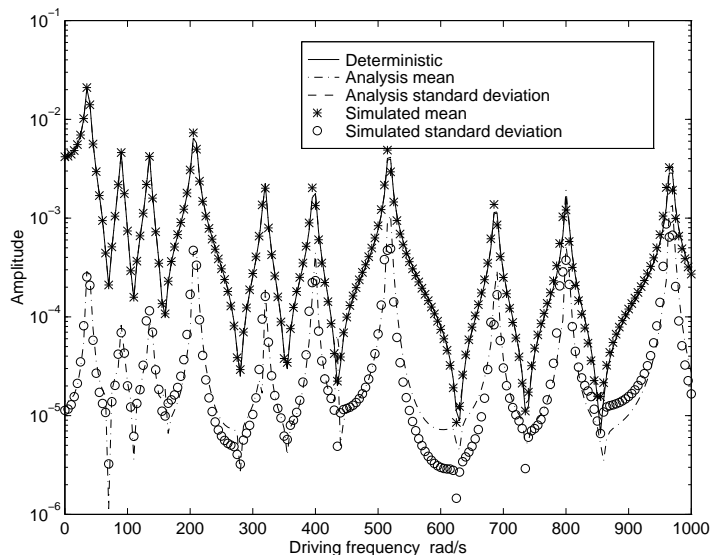


Figure 1.5: Statistics of amplitude of side sway using Neumann expansion method

tude and phase processes evolve in a nonstationary manner in  $\omega$ . The mean results are found to closely follow the deterministic results. However, in high frequency ranges the response variability increases, which gets characterized by relatively higher values of standard deviation, especially near the resonant frequency points, where the standard deviation sometimes becomes comparable to the mean value. This is significant, since, the standard deviations of the beam property random fields are only 5% of the corresponding mean values, which would mean that the system dynamics magnifies the structural uncertainties considerably in the high frequency ranges. This fact emphasizes the relevance of considering systems uncertainties, especially in higher driving frequency ranges.

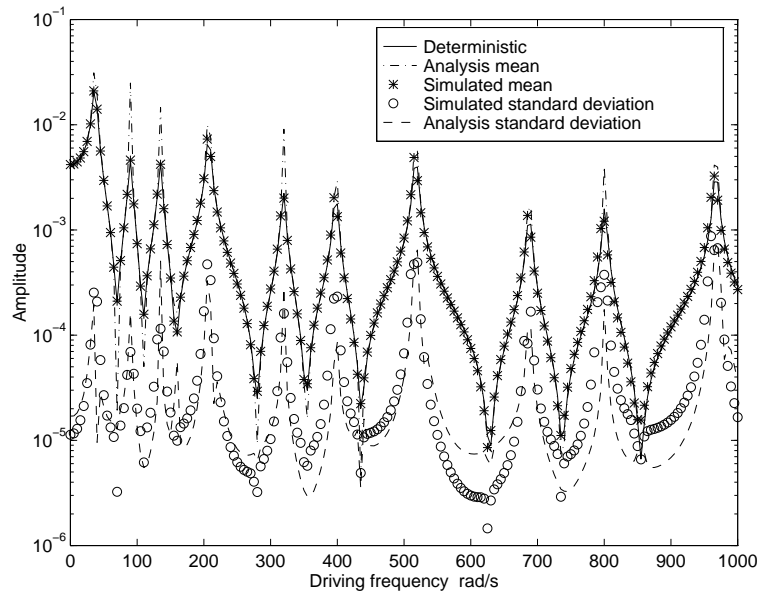


Figure 1.6: Statistics of amplitude of side sway using Combined Analytical-Simulation method

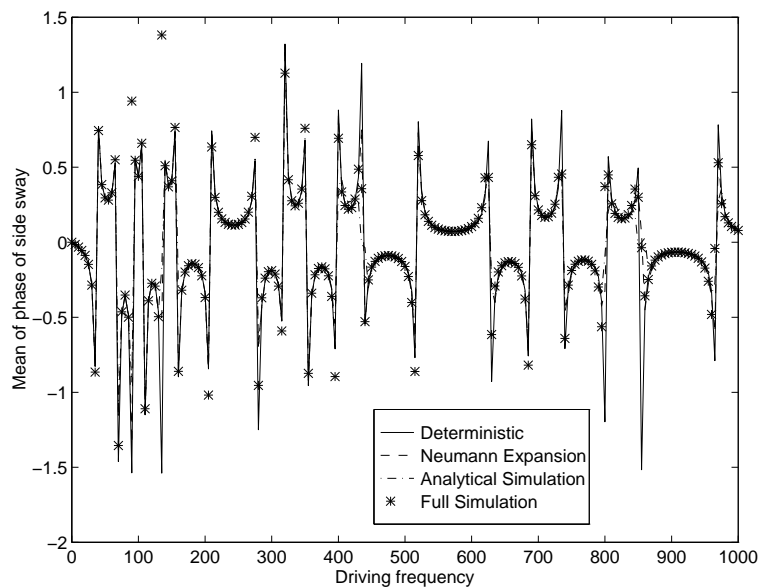


Figure 1.7: Mean of phase of side sway

## 1.5 Conclusions

The origins, quantification and propagation of uncertainties in structural dynamic models have been discussed in lecture 2. The differences between aleatoric and epistemic uncertainties have been pointed out and their potential sources were listed. Probabilistic model of uncertainty and their propagation methods using stochastic finite element method have been discussed. As an example harmonic vibration analysis of framed structures consisting of Euler-Bernoulli beam elements with stochastically inhomogeneous properties has been

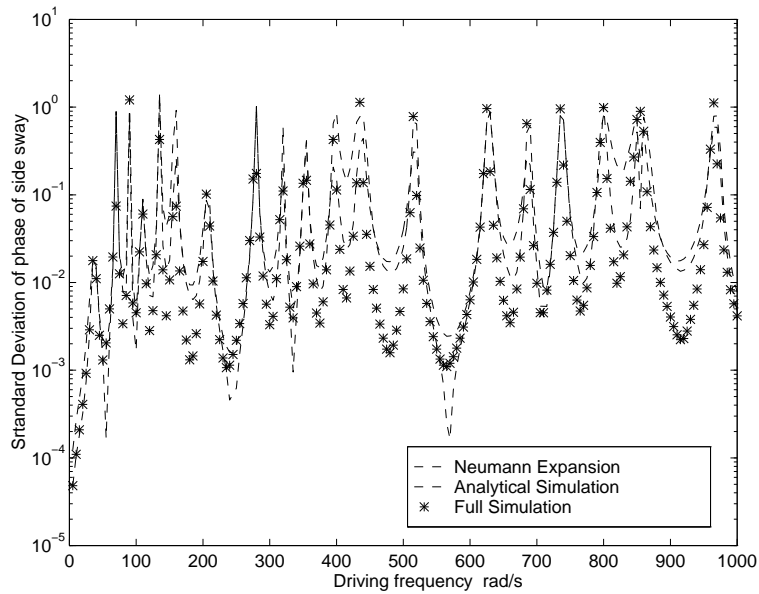


Figure 1.8: Standard deviation of phase of side sway

considered. The element property random fields were discretized using frequency dependent shape functions. This results in a complex random linear algebraic as the governing equation of motion after discretisation. Two analytical procedures and one combined analytical and simulation method are discussed for inverting this matrix and their performance is compared with the results of a more exact, but computationally intensive, full scale simulation scheme. These comparisons are demonstrated to be reasonably good over a wide range of driving frequency. The methods discussed herein are particularly advantageous if operating frequency range lies in higher system natural frequency ranges and also when several structural modes contribute to the response.

The approach presented here bypasses the need to solve the random eigenvalue problem. However, this method is limited to one dimensional elements with simple boundary conditions for which suitable frequency-dependent shape functions can be obtained. For more general class of problems, solution using the random eigensolutions is desirable. This topic is considered in the next section.

# Chapter 2

## Dynamic stiffness method using the Karhunen-Loève expansion

Uncertainties in complex dynamical systems play an important role in the prediction of dynamic response in the mid and high frequency ranges. For distributed parameter systems, parametric uncertainties can be represented by random fields leading to stochastic partial differential equations. Over the past two decades, the spectral stochastic finite element method has been developed to discretise the random fields and solve such problems. On the other hand, for deterministic distributed parameter linear dynamical systems, spectral finite element method has been developed to efficiently solve the problem in the frequency domain. In spite of the fact that both approaches use spectral decomposition (one for the random fields and while the other for the dynamic displacement fields), there has been very little overlap between them in literature. In this chapter these two spectral techniques have been unified with the aim that the unified approach would outperform any of the spectral methods considered on its own. Considering exponential autocorrelation function for the random fields, frequency dependent stochastic element stiffness and mass matrices are derived for axial and bending vibration of rods. Closed-form exact expressions are derived using the Karhunen-Loève expansion. Numerical examples are given to illustrate the unified spectral approach.

### 2.1 Introduction

Spectral methods are widely used in various branches of science and engineering. Due to their general nature, the meaning of spectral methods can be very different depending on the applications and the disciplines. In spite of these differences, the unifying factor between the spectral methods in different disciplines is that generally they are very powerful tools for the analytical and experimental treatments of wide ranging physical problems. In the context of the stochastic finite element method (see for example Shinozuka and Yamazaki, 1988, Ghanem and Spanos, 1991, Kleiber and Hien, 1992, Matthies *et al.*, 1997, Manohar and Adhikari, 1998a,b, Adhikari and Manohar, 1999, 2000, Haldar and Mahadevan, 2000, Sudret



and Der-Kiureghian, 2000, Elishakoff and Ren, 2003), spectral methods have been used extensively to analytically represent the random fields describing parametric uncertainties of physical systems. In particular, we refer to the recent chapter by Nouy (2009). In the context of structural dynamics, spectral methods have been used in random vibration problems (see for example Nigam, 1983, Lin, 1967, Bolotin, 1984) and for the discretisation of displacement fields in the frequency domain (Doyle, 1989, Gopalakrishnan *et al.*, 2007). In spite of the fact that both approaches use spectral decomposition (one for the random fields and while the other for the dynamic displacement fields), there has been very little overlap between them in literature. In this chapter these two spectral techniques have been unified with the aim that the unified approach would perform better than any of the spectral methods considered on its own.

In this chapter we focus our attention to structural dynamical systems with parametric uncertainties. Uncertainties should be taken into account for credible prediction of numerical codes. In the parametric approach, the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element method. The effect of uncertainty is significant in the higher frequency ranges. In the higher frequency ranges, as the wavelengths become smaller, very fine (static) mesh size is required to capture the dynamical behaviour. As a result, the deterministic analysis itself can pose significant computational challenges. One way to address this problem is to use a spectral approach in the frequency domain. The main idea here is that the displacements within an element are expressed in terms of frequency dependent shape functions. The shape functions adapt themselves with increasing frequency and consequently displacements can be obtained accurately without fine remeshing. The spectral approach has the potential to be an efficient method for mid and high frequency vibration problems provided the random fields describing parametric uncertainties can be taken into account efficiently. Here the spectral decomposition of the random files is used in conjunction with the spectral decomposition of the displacements field. It is expected that simultaneous use of these two types of spectral decomposition will result in an efficient approach for distributed dynamical systems with parametric uncertainties.

The outline of the chapter is as follows. Spectral finite element method in the frequency domain is briefly discussed in section 2.2. The essential background of spectral representation of stochastic fields is given in section 2.3. The general derivation of the element mass, stiffness and damping matrices using the doubly spectral stochastic finite element method is given in section 2.4. In section 2.5 this general theory is applied to axially vibrating rods with uncertain properties. The method is further applied to bending vibration of Euler-Bernoulli beams with random properties in section 2.6. Finally, some conclusions are drawn in section 2.7 based on the study undertaken in the chapter.

## 2.2 Spectral finite element method in the frequency domain

Spectral methods for deterministic dynamical systems have been in use for more than three decades (see for example the book by Paz (1980)). This approach, or approaches very similar to this, is known by various names such as the dynamic stiffness method (Banerjee and Williams, 1985, Banerjee, 1989, Banerjee and Williams, 1992, Banerjee and Fisher, 1992, Ferguson and Pilkey, 1993a,b, Banerjee and Williams, 1995, Banerjee, 1997), spectral finite element method (Doyle, 1989, Gopalakrishnan *et al.*, 2007) and dynamic finite element method (Hashemi *et al.*, 1999, Hashemi and Richard, 2000). Some of the notable features of the method are

1. the mass distribution of the element is treated in an exact manner in deriving the element dynamic stiffness matrix;
2. the dynamic stiffness matrix of one dimensional structural elements taking into account the effects of flexure, torsion, axial motion, shear deformation effects and damping are exactly determinable, which, in turn, enables the exact vibration analysis of skeletal structures by an inversion of the global dynamic stiffness matrix;
3. the method does not employ eigenfunction expansions and, consequently, a major step of the traditional finite element analysis, namely, the determination of natural frequencies and mode shapes, is eliminated which automatically avoids the errors due to series truncation; this makes the method attractive for situations in which a large number of modes participate in vibration;
4. since the modal expansion is not employed, ad hoc assumptions concerning damping matrix being proportional to mass and/or stiffness are not necessary;
5. the method is essentially a frequency domain approach suitable for steady state harmonic or stationary random excitation problems; generalization to other type of problems such as aeroelastic problems and dynamics of laminate composite materials through the use of Laplace and Fourier transforms is also available (Gopalakrishnan *et al.*, 2007);
6. the static stiffness matrix and the consistent mass matrix appear as the first two terms in the Taylor expansion of the dynamic stiffness matrix in the frequency parameter.

## 2.3 Spectral finite element method for stochastic field problem

Problems of structural dynamics in which the uncertainty in specifying stiffness and mass of the structure is modeled within the framework of random fields can be treated using the

stochastic finite element method (Ghanem and Spanos, 1991, Sudret and Der-Kiureghian, 2000, ?, Nouy, 2009). The application of the stochastic finite element method to linear structural dynamics problems typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions.
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the ‘discretisation of random fields’ is a major step in the analysis.
3. Formulation of the system equations of motion of the form  $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$  where  $\mathbf{D}(\omega)$  is the random dynamic stiffness matrix  $\mathbf{u}$  is the vector of random nodal displacement and  $\mathbf{f}$  is the vector of applied forces. In general  $\mathbf{D}(\omega)$  is a random symmetric complex matrix.
4. Solution of the set of complex random algebraic equation to obtain the statistics of the response vectors. Alternatively the response statistics can be obtained by solving the underlying random eigenvalue problem (see for example, Scheidt and Purkert (1983), Adhikari and Friswell (2007), Benaroya (1992), Adhikari (2007) and references therein).

We consider  $(\Theta, \mathcal{F}, P)$  be a probability space with  $\theta \in \Theta$  denoting a sampling point in the sampling space  $\Theta$ ,  $\mathcal{F}$  is the complete  $\sigma$ -algebra over the subsets of  $\Theta$  and  $P$  is the probability measure. Suppose the spatial coordinate vector  $\mathbf{r} \in \mathbb{R}^d$  where  $d \in \mathcal{I} \leq 3$  is the spatial dimension of the problem. Consider  $H : (\mathbb{R}^d \times \Theta) \rightarrow \mathbb{R}$  is a random field with a covariance function  $C_H : (\mathbb{R}^d \times \mathbb{R}^d) \rightarrow \mathbb{R}$  defined in a space  $\mathcal{D} \in \mathbb{R}^d$ . Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random process  $H(\mathbf{r}, \theta)$  can be expressed in a generalized Fourier type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j(\theta) \varphi_j(\mathbf{r}) \quad (2.1)$$

where  $\xi_j(\theta)$  are uncorrelated random variables,  $\lambda_j$  and  $\varphi_j(\mathbf{r})$  are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_H(\mathbf{r}_1, \mathbf{r}_2) \varphi_j(\mathbf{r}_1) d\mathbf{r}_1 = \lambda_j \varphi_j(\mathbf{r}_2), \quad \forall j = 1, 2, \dots \quad (2.2)$$

The spectral decomposition in Eq. (2.1) is known as the Karhunen-Loève expansion. The series in (2.1) can be ordered in a decreasing series so that it can be truncated using a finite number of terms with a desired accuracy. We refer the books by Ghanem and Spanos (1991), Papoulis and Pillai (2002) and references therein for further discussions on Karhunen-Loève expansion.

In this chapter one dimensional systems are considered. Moreover, Gaussian random fields with exponentially decaying autocorrelation function are considered. The autocorrelation function can be expressed as

$$C(x_1, x_2) = e^{-c|x_1-x_2|} \quad (2.3)$$

Here the quantity  $1/c$  is proportional to the correlation length and it plays an important role in the description of a random field. If the correlation length is very small, then the random process becomes close to a delta-correlated process, often known as the white noise. If the correlation length is very large compared to domain under consideration, then the random process effectively becomes a random variable. The underlying random process  $H(x, \theta)$  can be expanded using the Karhunen-Loève expansion (Ghanem and Spanos, 1991, Papoulis and Pillai, 2002) in the interval  $-l \leq x \leq l$  as

$$H(x, \theta) = \sum_{j=1}^{\infty} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x) \quad (2.4)$$

Since  $H(x, \theta)$  is assumed to be a Gaussian random variable, without any loss of generality we assumed the mean is zero in Eq. (2.4). The eigenvalues and eigenfunctions for odd  $j$  are given by

$$\lambda_j = \frac{2c}{\alpha_j^2 + c^2}, \quad \varphi_j(x) = \frac{\cos(\alpha_j x)}{\sqrt{l + \frac{\sin(2\alpha_j l)}{2\alpha_j}}}, \quad \text{where } \tan(\alpha_j l) = \frac{c}{\alpha_j}, \quad (2.5)$$

and for even  $j$  are given by

$$\lambda_j = \frac{2c}{\alpha_j^2 + c^2}, \quad \varphi_j(x) = \frac{\sin(\alpha_j x)}{\sqrt{l - \frac{\sin(2\alpha_j l)}{2\alpha_j}}}, \quad \text{where } \tan(\alpha_j l) = \frac{\alpha_j}{-c}. \quad (2.6)$$

These eigenvalues and eigenfunctions will be used to obtain the element mass, stiffness and damping matrices. For all practical purposes, the infinite series in Eq. (2.4) needs to be truncated using a finite number of terms. The number of terms could be selected based on the ‘amount of information’ to be retained. This in turn can be related to the number of eigenvalues retained, since the eigenvalues,  $\lambda_j$ , in Eq. (2.4) are arranged in decreasing order. For example, if 90% of the information is to be retained, then one can choose the number of terms,  $M$ , such that  $\lambda_M/\lambda_1 = 0.1$ . The value of  $M$  mainly depends on the correlation length of the underlying random field. One needs more terms when the correlation length is small. Intuitively this means that more independent variables are needed for fields with smaller correlation lengths and vice versa.

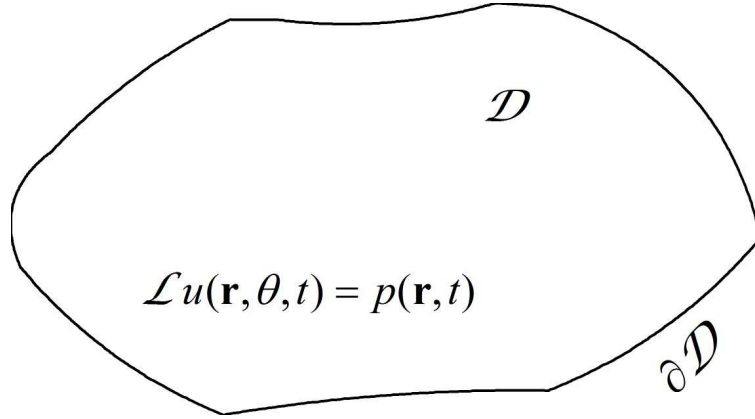


Figure 2.1: The domain and the boundary surface of the differential operator describing the stochastic dynamical system.

## 2.4 General derivation of doubly spectral element matrices

A linear damped distributed parameter dynamical system in which the displacement variable  $U(\mathbf{r}, t)$ , where  $\mathbf{r} \in \mathbb{R}^d$  is the spatial position vector,  $d \leq 3$  is the dimension of the model and  $t$  is time, specified in some domain  $\mathcal{D}$  as shown in Figure 2.1, is governed by a linear partial differential equation (Meirovitch, 1997):

$$\rho(\mathbf{r}, \theta) \frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_1(\theta) \frac{\partial U(\mathbf{r}, t)}{\partial t} + L_2(\theta) U(\mathbf{r}, t) = p(\mathbf{r}, t); \quad \mathbf{r} \in \mathcal{D}, t \in T \quad (2.7)$$

with linear boundary-initial conditions of the form

$$M_{1j} \frac{\partial U(\mathbf{r}, t)}{\partial t} = 0; \quad M_{2j} U(\mathbf{r}, t) = 0; \quad \mathbf{r} \in \partial\mathcal{D}, t = t_0, j = 1, 2, \dots \quad (2.8)$$

specified on some boundary surface  $\partial\mathcal{D}$ . In the above equation  $T \in \mathbb{R}$  is the domain of the time variable  $t$ ,  $\rho(\mathbf{r}, \theta)$  is the random mass distribution of the system,  $p(\mathbf{r}, t)$  is the distributed time-varying forcing function,  $L_1$  is the random spatial self-adjoint damping operator,  $L_2$  is the random spatial self-adjoint stiffness operator and  $M_{1j}$  and  $M_{2j}$  are some linear operators defined on the boundary surface  $\partial\mathcal{D}$ . When parametric uncertainties are considered, the mass density  $\rho(\mathbf{r}, \theta) : (\mathbb{R}^d \times \Theta) \rightarrow \mathbb{R}$  as well as the damping and stiffness operators involve random processes. Frequency dependent random element stiffness matrices were derived by various authors using the dynamic weighted integral approach (Manohar and Adhikari, 1998a,b, Adhikari and Manohar, 2000, Gupta and Manohar, 2002), the energy operator approach (Ghanem and Sarkar, 2003), sub-structure approach (Sarkar and Ghanem, 2003) and a series expansion approach (Ostoja-Starzewski and Woods, 2003). In the context of uncertainty modeling using Fuzzy approach, Nunes *et al.* (2006) have combined Fuzzy sets with the spectral approach. Moens and Vandepitte (2005), De Gerssem *et al.* (2005), Giannini and Hanss (2008), Moens and Vandepitte (2007) have used Fuzzy parametric approach for uncertainty quantification in the dynamic response.

Xiu and Karniadakis (2002, 2003), Wan and Karniadakis (2006) have proposed generalized polynomial chaos approach which can be used for spectral decomposition of random displacement fields. The method proposed here is motivated by the energy operator approach proposed by Sarkar and Ghanem (2002), Ghanem and Sarkar (2003) for the probabilistic case and the spectral approach proposed by Nunes *et al.* (2006) for Fuzzy uncertain variables. While numerical methods were used in these studies, in this chapter exact closed-form analytical expressions will be derived for the element matrices. Suppose the underlying homogeneous system corresponding to system (2.7) without any forcing (see for example Meirovitch, 1997) is given by

$$\rho_0 \frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_{10} \frac{\partial U(\mathbf{r}, t)}{\partial t} + L_{20} U(\mathbf{r}, t) = 0; \quad \mathbf{r} \in \mathcal{D} \quad (2.9)$$

together with suitable homogeneous boundary and initial conditions. Equation (2.9) is a deterministic equation. Taking the Fourier transform of Eq. (2.9) and considering zero initial conditions one has

$$-\omega^2 \rho_0 u(\mathbf{r}, \omega) + i\omega L_{10} \{u(\mathbf{r}, \omega)\} + L_{20} \{u(\mathbf{r}, \omega)\} = 0 \quad (2.10)$$

where  $\omega \in [0, \Omega]$  is the frequency and  $\Omega \in \mathbb{R}$  denotes the maximum frequency.

Like the classical finite element method, suppose that frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$u_e(\mathbf{r}, \omega) = \mathbf{N}^T(\mathbf{r}, \omega) \hat{\mathbf{u}}_e(\omega) \quad (2.11)$$

Here  $\hat{\mathbf{u}}_e(\omega) \in \mathbb{C}^n$  is the nodal displacement vector and  $\mathbf{N}(\mathbf{r}, \omega) \in \mathbb{C}^n$ , the vector of frequency-dependent shape functions and  $n$  is the number of the nodal degrees-of-freedom. Suppose the  $s_j(\mathbf{r}, \omega) \in \mathbb{C}, j = 1, 2, \dots, m$  are the basis functions which exactly satisfy Eq. (2.10). Here  $m$  is the order of the ordinary differential Eq. (2.10). It can be shown that the shape function vector can be expressed as

$$\mathbf{N}(\mathbf{r}, \omega) = \mathbf{\Gamma}(\omega) \mathbf{s}(\mathbf{r}, \omega) \quad (2.12)$$

where the vector  $\mathbf{s}(\mathbf{r}, \omega) = \{s_j(\mathbf{r}, \omega)\}^T, \forall j = 1, 2, \dots, m$  and the complex matrix  $\mathbf{\Gamma}(\omega) \in \mathbb{C}^{n \times m}$  depends on the boundary conditions. The derivation of  $\mathbf{\Gamma}(\omega)$  for axial vibration of rods and bending vibration of beams are given in the next two sections.

Extending the weak-form of finite element approach to the complex domain, the frequency dependent  $n \times n$  complex random stiffness, mass and damping matrices can be obtained as

$$\mathbf{K}_e(\omega, \theta) = \int_{\mathcal{D}_e} k_s(\mathbf{r}, \theta) \mathcal{L}_2 \{\mathbf{N}(\mathbf{r}, \omega)\} \mathcal{L}_2 \{\mathbf{N}^T(\mathbf{r}, \omega)\} d\mathbf{r} \quad (2.13)$$

$$\mathbf{M}_e(\omega, \theta) = \int_{\mathcal{D}_e} \rho(\mathbf{r}, \theta) \mathbf{N}(\mathbf{r}, \omega) \mathbf{N}^T(\mathbf{r}, \omega) d\mathbf{r} \quad \text{and} \quad (2.14)$$

$$\mathbf{C}_e(\omega, \theta) = \int_{\mathcal{D}_e} c(\mathbf{r}, \theta) \mathcal{L}_1 \{\mathbf{N}(\mathbf{r}, \omega)\} \mathcal{L}_1 \{\mathbf{N}^T(\mathbf{r}, \omega)\} d\mathbf{r} \quad (2.15)$$

Where,  $(\bullet)^T$  denotes matrix transpose,  $k_s(\mathbf{r}, \theta) : (\mathbb{R}^d \times \Theta) \rightarrow \mathbb{R}$  is the random distributed stiffness parameter,  $\mathcal{L}_2\{\bullet\}$  is the strain energy operator,  $c(\mathbf{r}, \theta) : (\mathbb{R}^d \times \Theta) \rightarrow \mathbb{R}$  is the random distributed damping parameter and  $\mathcal{L}_1\{\bullet\}$  is the energy dissipation operator. The derivation of the element matrices follows a method similar to the conventional spectral stochastic finite element method (see for example Ghanem and Spanos (1991)). The main difference is that the real shape functions need to be replaced by equivalent complex shape functions given by Eq. (2.12). We refer to the chapters by Manohar and Adhikari (1998a), Adhikari and Manohar (1999) for further details including the derivation of the complex element matrices using energy principles. In the above equations  $\mathcal{D}_e \in \mathcal{D}$  is the domain of an element such that  $\mathcal{D} = \bigcup \dots \bigcup \mathcal{D}_e$  and  $\mathcal{D}_e \cap \mathcal{D}_{e'} = \emptyset, \forall e, e'$ . The random fields  $k_s(\mathbf{r}, \theta)$ ,  $\rho(\mathbf{r}, \theta)$  and  $c(\mathbf{r}, \theta)$  are expanded using the Karhunen-Loève expansion (2.1). Using finite number of terms, each of the complex element matrices can be expanded in a spectral series as

$$\mathbf{K}_e(\omega, \theta) = \mathbf{K}_{0e}(\omega) + \sum_{j=1}^{M_K} \xi_{K_j}(\theta) \mathbf{K}_{je}(\omega) \quad (2.16)$$

$$\mathbf{M}_e(\omega, \theta) = \mathbf{M}_{0e}(\omega) + \sum_{j=1}^{M_M} \xi_{M_j}(\theta) \mathbf{M}_{je}(\omega) \quad (2.17)$$

$$\text{and } \mathbf{C}_e(\omega, \theta) = \mathbf{C}_{0e}(\omega) + \sum_{j=1}^{M_C} \xi_{C_j}(\theta) \mathbf{C}_{je}(\omega) \quad (2.18)$$

Here the complex deterministic symmetric matrices, for example in the case of the stiffness matrix, can be obtained as

$$\mathbf{K}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s0}(\mathbf{r}) \mathcal{L}_2\{\mathbf{N}(\mathbf{r}, \omega)\} \mathcal{L}_2\{\mathbf{N}^T(\mathbf{r}, \omega)\} \mathbf{d}\mathbf{r} \quad \text{and} \quad (2.19)$$

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{K_j}} \int_{\mathcal{D}_e} \varphi_{K_j}(\mathbf{r}) \mathcal{L}_2\{\mathbf{N}(\mathbf{r}, \omega)\} \mathcal{L}_2\{\mathbf{N}^T(\mathbf{r}, \omega)\} \mathbf{d}\mathbf{r} \quad (2.20)$$

$$\forall j = 1, 2, \dots, M_K$$

The equivalent terms corresponding to the mass and damping matrices can also be obtained in a similar manner. Substituting the shape function from Eq. (2.12), into equations (2.19) and (2.20) one obtains

$$\mathbf{K}_{0e}(\omega) = \mathbf{\Gamma}(\omega) \tilde{\mathbf{K}}_{0e}(\omega) \mathbf{\Gamma}^T(\omega) \quad \text{and} \quad (2.21)$$

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{K_j}} \mathbf{\Gamma}(\omega) \tilde{\mathbf{K}}_{je}(\omega) \mathbf{\Gamma}^T(\omega); \quad \forall j = 1, 2, \dots, M_K \quad (2.22)$$

where

$$\tilde{\mathbf{K}}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s0}(\mathbf{r}) \mathcal{L}_2\{\mathbf{s}(\mathbf{r}, \omega)\} \mathcal{L}_2\{\mathbf{s}^T(\mathbf{r}, \omega)\} \mathbf{d}\mathbf{r} \in \mathbb{C}^{mm} \quad \text{and} \quad (2.23)$$

$$\tilde{\mathbf{K}}_{je}(\omega) = \int_{\mathcal{D}_e} \varphi_{K_j}(\mathbf{r}) \mathcal{L}_2\{\mathbf{s}(\mathbf{r}, \omega)\} \mathcal{L}_2\{\mathbf{s}^T(\mathbf{r}, \omega)\} \mathbf{d}\mathbf{r} \in \mathbb{C}^{mm} \quad (2.24)$$

$$\forall j = 1, 2, \dots, M_K$$

The expressions of the eigenfunctions given in the previous section are valid within the specific domains defined before. One needs to change the coordinate in order to use them in Eq. (2.24). Once the element stiffness, mass and damping matrices are obtained in this manner, the global matrices can be calculated by summing the element matrices with suitable coordinate transformations as in the standard finite element method. A closed-form expression of the eigenfunctions appearing in Eq. (2.24) are available for only few specific correlation functions and with simple boundaries only. For such cases, as will be seen later in the chapter, the integral in Eq. (2.24) may be obtained in closed-form. However, in general the integral equation governing the eigenfunctions in (2.2) has to be solved numerically. For such general cases the element matrices should be obtained using numerical integration techniques.

Due to the use of spectral element in the frequency domain, only one finite element is required per physical ‘element’ of a built-up system. For this reason, the dimension of the global assembled matrices become small even when high-frequency vibration is considered. However, the deterministic system, the element matrices are not exact as the Karhunen-Loève expansion (2.1) needs to be truncated after a finite number of terms. The global spectral matrix can be expressed as

$$\mathbf{D}(\omega, \theta) = -\omega^2 \mathbf{M}(\omega, \theta) + i\omega \mathbf{C}(\omega, \theta) + \mathbf{K}(\omega, \theta) \in \mathbb{C}^{N \times N} \quad (2.25)$$

where  $N$  is the dynamic degrees of freedom. Following the proposed DSSFEM approach, in general the matrix  $\mathbf{D}(\omega, \theta)$  can be expressed as

$$\mathbf{D}(\omega, \theta) = \mathbf{D}_0(\omega) + \sum_j \xi_j(\theta) \mathbf{D}_j(\omega) \quad (2.26)$$

In this equation  $\mathbf{D} : (\Omega \times \Theta) \rightarrow \mathbb{C}^{N \times N}$  is a complex random symmetric matrix and it needs to be inverted for every  $\omega$  to obtain the dynamic response. Here  $\Omega$  denotes the space of frequency. Unlike the inversion of real symmetric random matrices or complex Hermitian matrices, relatively less literature is available on complex symmetric matrices. Adhikari and Manohar (1999) and more recently Ghanem and Das (2009) have considered complex random matrices arising in structural dynamics. In principle analytical approaches such as the perturbation based methods (Kleiber and Hien, 1992) and projections methods (Ghanem and Spanos, 1991) can be applied for the inversion of  $\mathbf{D}(\omega, \theta)$ . In practice, however, difficulties may arise due to the fact that  $\mathbf{D}(\omega, \theta)$  becomes close to singular when  $\omega$  approaches to a system natural frequency. This can be a major problem particularly when the damping of the system is low. Reliable and computationally efficient methods for the derivation of dynamic response using the proposed DSSFEM approach is an outstanding problem and is currently a limitation of this approach. It is beyond the scope of this chapter to address this issue in details. Here direct Monte Carlo simulation is used to obtain the response statistics in the numerical examples to be followed.



## 2.5 DSSFEM for damped rods in axial vibration

### 2.5.1 The equation of motion

The equation of motion of a damped stochastically inhomogeneous rod under axial vibration is given by

$$\frac{\partial}{\partial x} \left[ AE(x) \frac{\partial U(x, t)}{\partial x} + c_1 \frac{\partial^2 U(x, t)}{\partial x \partial t} \right] = m(x) \frac{\partial^2 U(x, t)}{\partial t^2} + c_2 \frac{\partial U(x, t)}{\partial t} \quad (2.27)$$

Here  $U(x, t)$  is the axial displacement,  $c_1$  is the strain rate dependent viscous damping coefficient and  $c_2$  is the velocity dependent viscous damping coefficient. These quantities are assumed to be deterministic constants. The axial rigidity  $AE(x)$  and the mass per unit length  $m(x)$  are assumed to be random fields of the following form

$$AE(x, \theta) = AE_0 [1 + \epsilon_{AE} H_{AE}(x, \theta)] \quad (2.28)$$

$$m(x) = m_0 (1 + \epsilon_m H_m(x, \theta)) \quad (2.29)$$

It is assumed that  $H_{AE}(x, \theta)$  and  $H_m(x, \theta)$  are homogeneous Gaussian random fields with zero mean and exponentially decaying autocorrelation function of the form given by Eq. (2.3). The ‘strength parameters’  $\epsilon_{AE}$  and  $\epsilon_m$  effectively quantify the amount of uncertainty in the axial rigidity and mass per unit length of the rod. The constants  $AE_0$  and  $m_0$  are respectively the mass per unit length and axial rigidity of the underlying baseline model. The equation of motion of the baseline model is given by

$$AE_0 \frac{\partial^2 U(x, t)}{\partial x^2} + c_1 \frac{\partial^3 U(x, t)}{\partial x^2 \partial t} = m_0 \frac{\partial^2 U(x, t)}{\partial t^2} + c_2 \frac{\partial U(x, t)}{\partial t} \quad (2.30)$$

With the spectral expansion of the axial displacement  $U(x, t)$  in the frequency-wavenumber space, one has

$$U(x, t) = u(x) e^{i\omega t} = e^{kx} e^{i\omega t} \quad (2.31)$$

and  $k$  is the wavenumber for the baseline model in Eq. (2.30). Substituting  $U(x, t)$  from Eq. (2.31) in Eq. (2.30) and simplifying we have

$$k^2 + a^2 = 0 \quad \text{or} \quad k = \pm ia \quad (2.32)$$

where

$$a^2 = \frac{m_0 \omega^2 - i\omega c_2}{AE_0 + i\omega c_1} \quad (2.33)$$

An element for the damped axially vibrating rod is shown in Figure 2.2. In view of the solutions in Eq. (2.32), the complex displacement field within the element can be expressed by linear combination of the basic functions  $e^{-iax}$  and  $e^{iax}$  so that in our notations  $\mathbf{s}(x, \omega) = \{e^{-iax}, e^{iax}\}^T$ . We have expressed the KL expansion in terms of trigonometric functions in Eqs. (2.5) and (2.6). Therefore, it is more convenient to express  $\mathbf{s}(x, \omega)$  in

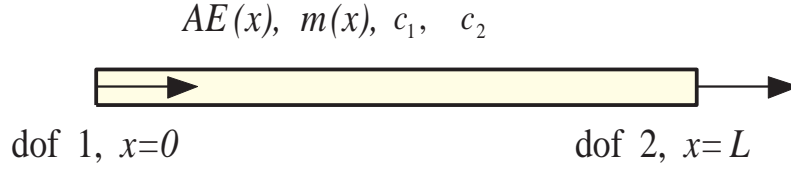


Figure 2.2: An element for the axially vibrating rod with damping. The axial rigidity  $AE(x)$  and mass per unit length  $m(x)$  are assumed to be random fields. The strain rate dependent viscous damping coefficient  $c_1$  and the velocity dependent viscous damping coefficient  $c_2$  are assumed to be deterministic. The element has two degrees of freedom and the displacement field within the element is complex and frequency dependent.

terms of trigonometric functions. Considering  $e^{\pm iax} = \cos(ax) \pm i \sin(ax)$ , the vector  $\mathbf{s}(x, \omega)$  can be alternatively expressed as

$$\mathbf{s}(x, \omega) = \begin{Bmatrix} \sin(ax) \\ \cos(ax) \end{Bmatrix} \in \mathbb{C}^2 \quad (2.34)$$

Considering unit axial displacement boundary condition as  $u(x=0) = 1$  and  $u(x=L) = 1$ , after some elementary algebra, the shape function vector can be expressed in the form of Eq. (2.12) as

$$\mathbf{N}(x, \omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(x, \omega), \quad \text{where} \quad \mathbf{\Gamma}(\omega) = \begin{bmatrix} -\cot(aL) & 1 \\ \operatorname{cosec}(aL) & 0 \end{bmatrix} \in \mathbb{C}^{2 \times 2} \quad (2.35)$$

Now we need to substitute  $\mathbf{s}(x, \omega)$  in Eq. (2.23) and (2.24) to obtain the deterministic and random part of the element matrices. In this chapter damping is assumed to be deterministic. Therefore, only the stiffness and mass matrices of the system will be derived.

## 2.5.2 Derivation of the element stiffness and mass matrices

For the axial vibration, the stiffness operator is given by  $\mathcal{L}_2(\bullet) = \frac{\partial(\bullet)}{\partial x}$ . Because constant nominal values are assumed, we have  $k_{s_0}(\mathbf{r}) = AE_0$ . Using these, from Eq. (2.23) one obtains

$$\tilde{\mathbf{K}}_{0e}(\omega) = AE_0 \int_{x=0}^L \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\} \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\}^T dx \quad (2.36)$$

$$= \frac{AE_0 a}{2} \begin{bmatrix} cs + aL & -1 + c^2 \\ -1 + c^2 & aL - cs \end{bmatrix} \quad (2.37)$$

where

$$c = \cos(aL) \quad \text{and} \quad s = \sin(aL) \quad (2.38)$$

The deterministic part of the stiffness matrix can be obtained from Eq. (2.21) using the  $\mathbf{\Gamma}(\omega)$  matrix defined in Eq. (2.35). The term  $\tilde{\mathbf{M}}_{0e}(\omega)$  can be obtained in a similar way as

$$\tilde{\mathbf{M}}_{0e}(\omega) = m_0 \int_{x=0}^L \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \quad (2.39)$$

$$= \frac{m_0}{2a} \begin{bmatrix} aL - cs & 1 - c^2 \\ 1 - c^2 & cs + aL \end{bmatrix} \quad (2.40)$$

The deterministic mass matrix can be obtained from the above equation as  $\mathbf{M}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\widetilde{\mathbf{M}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega)$ .

To obtain the matrices associated with the random components, note that for each  $j$  there will be two different matrices corresponding to the two eigenfunctions defined in equations (2.5) and (2.6). Following Eq. (2.16), we can express the element stiffness matrix as

$$\mathbf{K}_e(\omega, \theta) = \mathbf{K}_{0e}(\omega) + \Delta\mathbf{K}_e(\omega, \theta) \quad (2.41)$$

where  $\Delta\mathbf{K}_e(\omega, \theta)$  is the random part of the matrix. Following Eq. (2.22), this matrix can be conveniently expressed as

$$\Delta\mathbf{K}_e(\omega, \theta) = \mathbf{\Gamma}(\omega)\widetilde{\Delta\mathbf{K}}_e(\omega, \theta)\mathbf{\Gamma}^T(\omega) \quad (2.42)$$

The matrix  $\widetilde{\Delta\mathbf{K}}_e(\omega)$  can be expanded utilizing the Karhunen-Loève expansion as

$$\widetilde{\Delta\mathbf{K}}_e(\omega, \theta) = \sum_{j=1}^{M_K} \xi_{K_j}(\theta) \sqrt{\lambda_{K_j}} \widetilde{\mathbf{K}}_{je}(\omega) \quad (2.43)$$

where  $\sqrt{\lambda_{K_j}}$  are the eigenvalues corresponding to the random field  $H_{AE}(x, \theta)$ . The matrices  $\widetilde{\mathbf{K}}_{je}(\omega)$  can be obtained using the integrals of Eq. (2.24). Using the expression of the eigenfunction for the odd values of  $j$  as in Eq. (2.5) one has

$$\begin{aligned} \widetilde{\mathbf{K}}_{je}(\omega) &= \int_0^L \frac{\epsilon_{AE}AE_0 \cos(\alpha_j(-L/2 + x))}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\} \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\}^T dx \\ &= \frac{\epsilon_{AE}AE_0}{\sqrt{(L/2 + c_\alpha s_\alpha / \alpha_j)} \alpha_j (4a^2 - \alpha_j^2)} \times \\ &\quad \begin{bmatrix} 2\alpha_j a c_\alpha cs + (-\alpha_j^2 + 4a^2 - \alpha_j^2 c^2) s_\alpha & (-2\alpha_j a + 2\alpha_j a c^2) c_\alpha + \alpha_j^2 s_\alpha cs \\ (-2\alpha_j a + 2\alpha_j a c^2) c_\alpha + \alpha_j^2 s_\alpha cs & -2\alpha_j a c_\alpha cs + (4a^2 - \alpha_j^2 + \alpha_j^2 c^2) s_\alpha \end{bmatrix} \end{aligned} \quad (2.44)$$

In the above expression

$$c_{\alpha_j} = \cos(\alpha_j L/2) \quad \text{and} \quad s_{\alpha_j} = \sin(\alpha_j L/2) \quad (2.46)$$

and the eigenvalues  $\alpha_j$  should be obtained by solving the transcendental Eq. (2.5) with  $l = L/2$ . In Eq. (2.44) the KL eigenfunction is shifted to take account of the fact that Eq. (2.5) is defined for  $-L/2 \leq x \leq L/2$  while the element shape functions are defined over  $0 \leq x \leq L$ . In Eq. (2.44) we have used the identity  $\sin(\alpha_j L) = 2 \cos(\alpha_j L/2) \sin(\alpha_j L/2) = 2c_\alpha s_\alpha$ . In a similar manner, using the expression of the eigenfunction for the even values of  $j$  as in Eq. (2.6) one has

$$\widetilde{\mathbf{K}}_{je}(\omega) = \int_0^L \frac{\epsilon_{AE}AE_0 \sin(\alpha_j(-L/2 + x))}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\} \left\{ \frac{\partial \mathbf{s}(x, \omega)}{\partial x} \right\}^T dx \quad (2.47)$$

$$\begin{aligned}
&= \frac{\epsilon_{AE}AE_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \frac{a^2}{\alpha_j (4a^2 - \alpha_j^2)} \times \\
&\left[ \begin{array}{cc} (-\alpha_j^2 + \alpha_j^2 c^2) c_{\alpha_j} + 2\alpha_j a s_{\alpha_j} c s & -\alpha_j^2 c_{\alpha_j} c s + 2\alpha_j a s_{\alpha_j} c^2 \\ -\alpha_j^2 c_{\alpha_j} c s + 2\alpha_j a s_{\alpha_j} c^2 & (\alpha_j^2 - \alpha_j^2 c^2) c_{\alpha_j} - 2\alpha_j a s_{\alpha_j} c s \end{array} \right] \quad (2.48)
\end{aligned}$$

The mass matrix can also be represented as equations (2.41)–(2.43). The eigenvalues and eigenfunctions corresponding to the random field  $H_m(x, \theta)$  needs to be used to obtain the elements of  $\widetilde{\mathbf{M}}_{je}(\omega)$ . Using the expression of the eigenfunction for the odd values of  $j$  as in Eq. (2.5) one has

$$\begin{aligned}
\widetilde{\mathbf{M}}_{je}(\omega) &= \int_0^L \frac{\epsilon_m m_0 \cos(\alpha_j(-L/2 + x))}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \quad (2.49) \\
&= \frac{\epsilon_m m_0}{\sqrt{(L/2 + c_\alpha s_\alpha/\alpha_j)}} \frac{1}{\alpha_j (4a^2 - \alpha_j^2)} \times \\
&\left[ \begin{array}{cc} -2\alpha_j a c_{\alpha_j} c s + (4a^2 - \alpha_j^2 + \alpha_j^2 c^2) s_{\alpha_j} & (2\alpha_j a - 2\alpha_j a c^2) c_{\alpha_j} - \alpha_j^2 s_{\alpha_j} c s \\ (2\alpha_j a - 2\alpha_j a c^2) c_{\alpha_j} - \alpha_j^2 s_{\alpha_j} c s & 2\alpha_j a c_{\alpha_j} c s + (-\alpha_j^2 + 4a^2 - \alpha_j^2 c^2) s_{\alpha_j} \end{array} \right] \quad (2.50)
\end{aligned}$$

In the above expression the eigenvalues  $\alpha_j$  should be obtained by solving the transcendental Eq. (2.5). In a similar manner, using the expression of the eigenfunction for the even values of  $j$  as in Eq. (2.6) one has

$$\begin{aligned}
\widetilde{\mathbf{M}}_{je}(\omega) &= \int_0^L \frac{\epsilon_m m_0 \sin(\alpha_j(-L/2 + x))}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \quad (2.51) \\
&= \frac{\epsilon_m m_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \frac{1}{\alpha_j (4a^2 - \alpha_j^2)} \times \\
&\left[ \begin{array}{cc} (\alpha_j^2 - \alpha_j^2 c^2) c_{\alpha_j} - 2\alpha_j a s_{\alpha_j} c s & \alpha_j^2 c_{\alpha_j} c s - 2\alpha_j a s_{\alpha_j} c^2 \\ \alpha_j^2 c_{\alpha_j} c s - 2\alpha_j a s_{\alpha_j} c^2 & (-\alpha_j^2 + \alpha_j^2 c^2) c_{\alpha_j} + 2\alpha_j a s_{\alpha_j} c s \end{array} \right] \quad (2.52)
\end{aligned}$$

Equations (2.44)–(2.51) completely define the random parts of the element stiffness and mass matrices. The exact closed-form expression of the elements of the above four matrices further reduces the computational cost in deriving these matrices.

### 2.5.3 Numerical Illustrations

We consider a numerical example to illustrate the application of the expressions derived in the previous subsection. The mean material properties are considered  $\rho_0 = 2700\text{kg/m}^3$  and  $E_0 = 69\text{GPa}$ , values corresponding to aluminium. Length and cross section of the rod are respectively  $L = 30\text{m}$  and  $A_0 = 1\text{cm}^2$ . Using these we have  $AE_0 = 6.9 \times 10^6$  and  $m_0 = \rho_0 A_0 = 0.27$ . A clamped-free boundary condition is considered. The standard deviations of both the random fields are assumed to be 10% of the mean values of the random fields, that is,  $\epsilon_{AE} = 0.1AE_0$  and  $\epsilon_m = 0.1m_0$ . The damping coefficients are assumed to be  $c_1 = 1.5 \times 10^{-5}AE_0$  and  $c_2 = 11.15m_0$ . The correlation length of the random fields describing

$AE(x)$  and  $m(x)$  is assumed to be  $L/5$ . We consider the response at the free end of the rod due to unit harmonic force at that end. The response is calculated upto 500Hz covering the first six vibration modes of the system. The response of the deterministic system, the mean and the standard deviation of the absolute value of the response are shown in Figure 2.3. These results are obtained using Monte Carlo simulation with 4000 samples. In total 36 terms

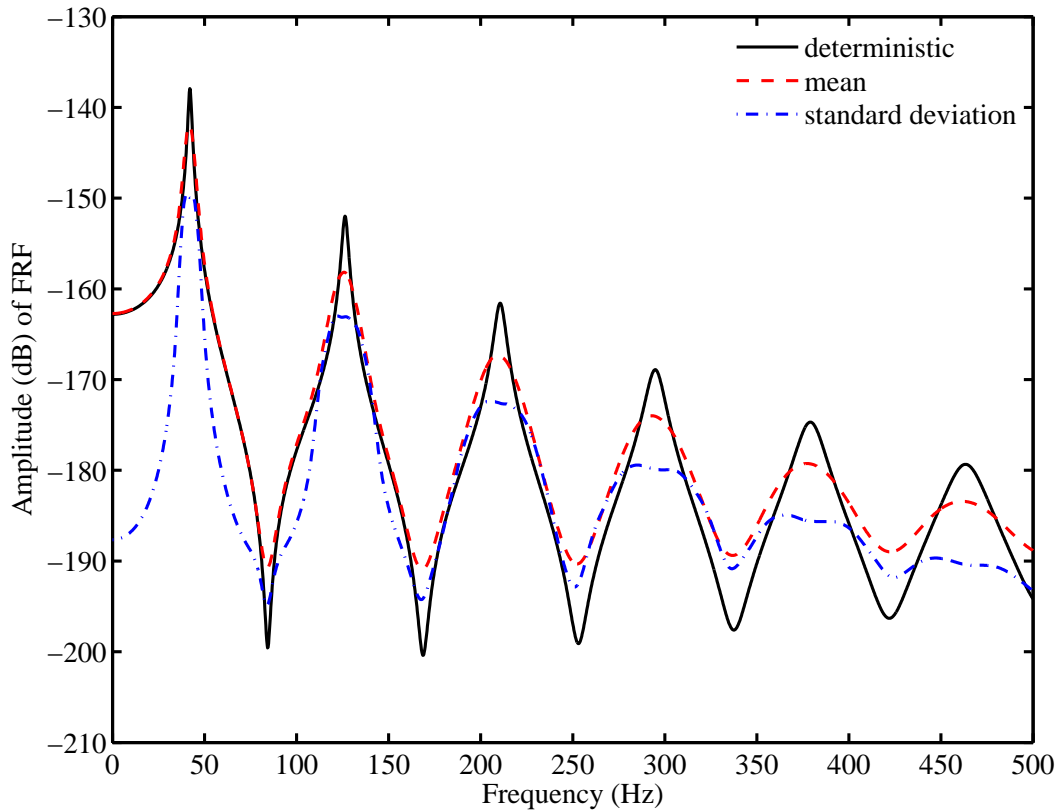


Figure 2.3: The amplitude of the frequency response function at the free end of the damped axially vibrating rod with random properties.

are used for the KL expansion. With this number of terms, the last eigenvalue of the KL expansion becomes less than 5% of the first eigenvalue. The element matrices associated with 36 random variables are obtained using the closed-form expression derived in the previous section. The phase of the frequency response function at the free end of the rod is shown in Figure 2.4. The phase does not change sign because we are considering the driving point response. In both Figs. 2.3 and 2.4 the mean curve is different from the deterministic curve. This difference is larger at higher frequencies. At lower frequencies, the standard deviation is biased by the mean. But as frequency increases, the standard deviation curve flattens. These results are obtained using a single spectral element although six modes of vibration exist within the frequency range considered.

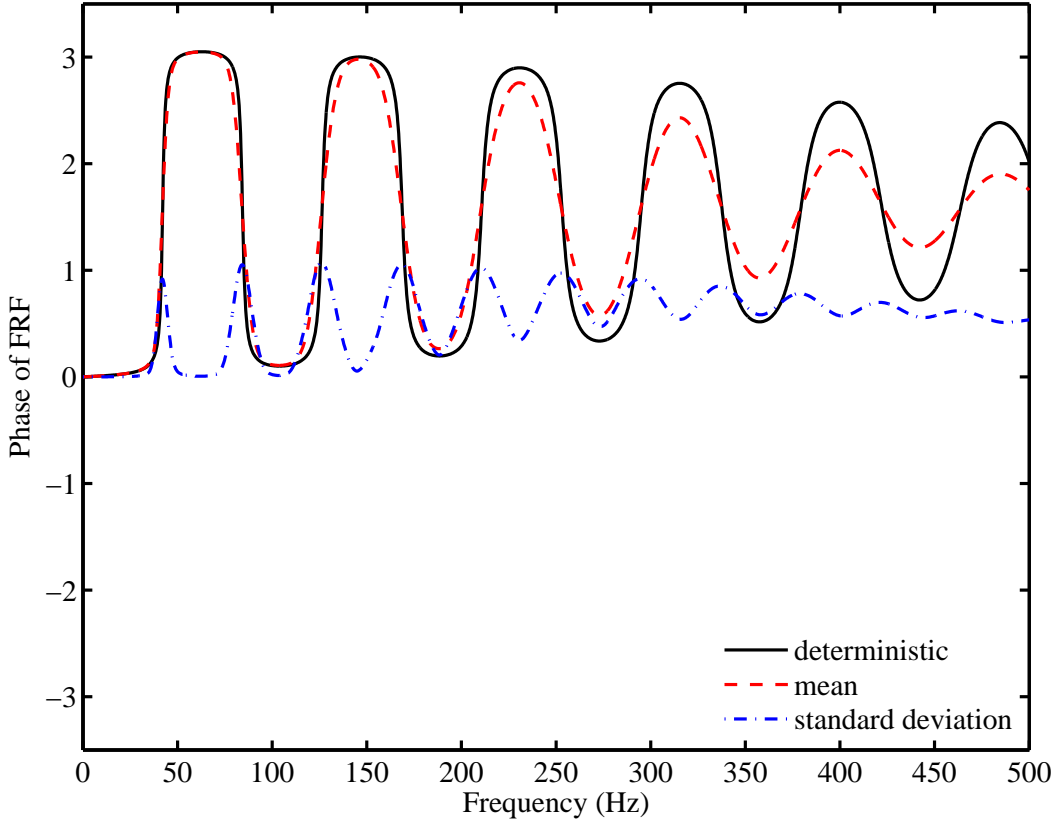


Figure 2.4: The phase of the frequency response function at the free end of the damped axially vibrating rod with random properties.

## 2.6 DSSFEM for damped beams in bending vibration

### 2.6.1 The equation of motion

The equation of motion of a damped stochastically inhomogeneous Euler-Bernoulli beam under bending vibration is given by

$$\frac{\partial^2}{\partial x^2} \left[ EI(x) \frac{\partial^2 Y(x, t)}{\partial x^2} + c_3 \frac{\partial^3 Y(x, t)}{\partial x^2 \partial t} \right] + m(x) \frac{\partial^2 Y(x, t)}{\partial t^2} + c_4 \frac{\partial Y(x, t)}{\partial t} = 0 \quad (2.53)$$

Here  $Y(x, t)$  is the transverse flexural displacement,  $c_3$  is the strain rate dependent viscous damping coefficient and  $c_4$  is the velocity dependent viscous damping coefficient. These quantities are assumed to be deterministic constants. The mass per unit length  $m(x)$  is assumed to be a random field of the form given by (2.29) and the bending rigidity  $EI(x)$  is assumed to be

$$EI(x, \theta) = EI_0 [1 + \epsilon_{EI} H_{EI}(x, \theta)] \quad (2.54)$$

Like the case of the axially vibrating rod, we consider that  $H_{EI}(x, \theta)$  is a homogeneous Gaussian random fields with zero mean and exponentially decaying autocorrelation function of the form given by Eq. (2.3). The ‘strength parameter’  $\epsilon_{EI}$  quantify the amount of

uncertainty in the bending rigidity of the beam. The constant  $EI_0$  is the bending rigidity of the underlying baseline model. The equation of motion of the baseline model is given by

$$EI_0 \frac{\partial^4 Y(x, t)}{\partial x^4} + c_3 \frac{\partial^5 Y(x, t)}{\partial x^2 \partial t} + m_0 \frac{\partial^2 Y(x, t)}{\partial t^2} + c_4 \frac{\partial Y(x, t)}{\partial t} = 0 \quad (2.55)$$

Using the spectral representation of the transverse displacement  $Y(x, t)$  one has

$$Y(x, t) = y(x) e^{i\omega t} = e^{kx} e^{i\omega t} \quad (2.56)$$

where  $k$  is the wavenumber for the baseline model in Eq. (2.55). Substituting  $Y(x, t)$  from Eq. (2.56) in Eq. (2.55) we have

$$k^4 - b^4 = 0 \quad \text{or} \quad k = \pm ib, \pm b \quad (2.57)$$

where

$$b^4 = \frac{m_0 \omega^2 - i\omega c_4}{EI_0 + i\omega c_3} \quad (2.58)$$

An element for the damped beam under bending vibration is shown in Figure 2.5. The degrees-of-freedom for each nodal point include a vertical and a rotational degrees-of-freedom. In view of the solutions in Eq. (2.57), the displacement field with the element can

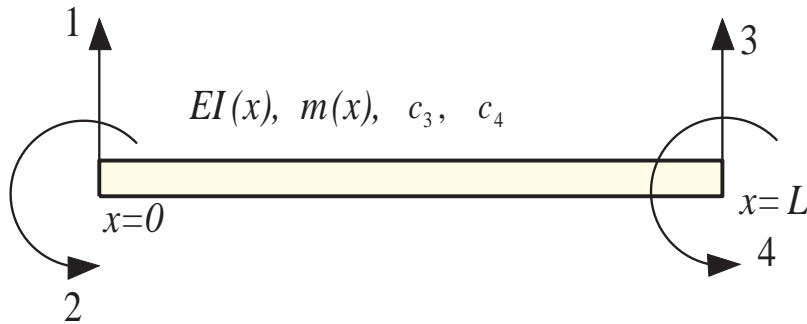


Figure 2.5: An element for the damped beam under bending vibration. The bending rigidity  $EI(x)$  and mass per unit length  $m(x)$  are assumed to be random fields. The strain rate dependent viscous damping coefficient  $c_3$  and the velocity dependent viscous damping coefficient  $c_4$  are assumed to be deterministic. The element has four degrees of freedom and the displacement field within the element is complex and frequency dependent.

be expressed by linear combination of the basic functions  $e^{-bx}$ ,  $e^{bx}$ ,  $e^{-ibx}$  and  $e^{ibx}$  so that in our notations  $\mathbf{s}(x, \omega) = \{e^{-bx}, e^{bx}, e^{-ibx}, e^{ibx}\}^T$ . We have expressed the KL expansion in terms of trigonometric functions in Eqs. (2.5) and (2.6). Therefore, like the previous section, we aim to express  $\mathbf{s}(x, \omega)$  in terms of trigonometric functions. Considering  $e^{\pm ibx} = \cos(bx) \pm i \sin(bx)$  and  $e^{\pm bx} = \cosh(bx) \pm i \sinh(bx)$ , the vector  $\mathbf{s}(x, \omega)$  can be alternatively expressed as

$$\mathbf{s}(x, \omega) = \begin{Bmatrix} \sin(bx) \\ \cos(bx) \\ \sinh(bx) \\ \cosh(bx) \end{Bmatrix} \in \mathbb{C}^4 \quad (2.59)$$

The displacement field within the element can be expressed as

$$y(x) = \mathbf{s}(x, \omega)^T \mathbf{y}_e \quad (2.60)$$

where  $\mathbf{y}_e \in \mathbb{C}^4$  is the vector of constants to be determined from the boundary conditions.

The relationship between the shape functions and the boundary conditions can be represented as in Table 2.1, where boundary conditions in each column give rise to the corresponding shape function. Writing Eq. (2.60) for the above four sets of boundary conditions,

	$N_1(x, \omega)$	$N_2(x, \omega)$	$N_3(x, \omega)$	$N_4(x, \omega)$
$y(0)$	1	0	0	0
$\frac{dy}{dx}(0)$	0	1	0	0
$y(L)$	0	0	1	0
$\frac{dy}{dx}(L)$	0	0	0	1

Table 2.1: The relationship between the boundary conditions and the shape functions for the bending vibration of beams.

one obtains

$$[\mathbf{R}] [\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4] = \mathbf{I} \quad (2.61)$$

where

$$\mathbf{R} = \begin{bmatrix} s_1(0) & s_2(0) & s_3(0) & s_4(0) \\ \frac{ds_1}{dx}(0) & \frac{ds_2}{dx}(0) & \frac{ds_3}{dx}(0) & \frac{ds_4}{dx}(0) \\ s_1(L) & s_2(L) & s_3(L) & s_4(L) \\ \frac{ds_1}{dx}(L) & \frac{ds_2}{dx}(L) & \frac{ds_3}{dx}(L) & \frac{ds_4}{dx}(L) \end{bmatrix} \quad (2.62)$$

and  $\mathbf{y}_e^k$  is the vector of constants giving rise to the  $k$ th shape function. In view of the boundary conditions represented in Table 2.1 and equation (2.61), the shape functions for bending vibration can be shown to be given by Eq. (2.12) where

$$\mathbf{\Gamma}(\omega) = [\mathbf{y}_e^1, \mathbf{y}_e^2, \mathbf{y}_e^3, \mathbf{y}_e^4]^T = [\mathbf{R}^{-1}]^T = \begin{bmatrix} \frac{1}{2} \frac{cS + Cs}{cC - 1} & -\frac{1}{2} \frac{1 + sS - cC}{cC - 1} & -\frac{1}{2} \frac{cS + Cs}{cC - 1} & \frac{1}{2} \frac{cC + sS - 1}{cC - 1} \\ \frac{1}{2} \frac{cC + sS - 1}{b(cC - 1)} & \frac{1}{2} \frac{-Cs + cS}{b(cC - 1)} & -\frac{1}{2} \frac{1 + sS - cC}{b(cC - 1)} & -\frac{1}{2} \frac{-Cs + cS}{b(cC - 1)} \\ -\frac{1}{2} \frac{S + s}{cC - 1} & \frac{1}{2} \frac{C - c}{cC - 1} & \frac{1}{2} \frac{S + s}{cC - 1} & -\frac{1}{2} \frac{C - c}{cC - 1} \\ \frac{1}{2} \frac{C - c}{b(cC - 1)} & -\frac{1}{2} \frac{S - s}{b(cC - 1)} & -\frac{1}{2} \frac{C - c}{b(cC - 1)} & -\frac{1}{2} \frac{S - s}{b(cC - 1)} \end{bmatrix} \quad (2.63)$$

Here

$$C = \cosh(bL), \quad c = \cos(bL), \quad S = \sinh(bL) \quad \text{and} \quad s = \sin(bL) \quad (2.64)$$

are frequency dependent quantities because  $b$  is a function of  $\omega$ . We need to substitute  $\mathbf{s}(x, \omega)$  in Eq. (2.23) and (2.24) to obtain the deterministic and random part of the element matrices. Since damping is assumed to be deterministic, we will only derive the stiffness and mass matrices of the system.



### 2.6.2 Derivation of the element stiffness and mass matrices

For the bending vibration the stiffness operator can be given as  $\mathcal{L}_2(\bullet) = \frac{\partial^2(\bullet)}{\partial x^2}$ . Because constant nominal values are assumed, we have  $k_{s_0}(\mathbf{r}) = EI_0$ . Using these, from Eq. (2.23) one obtains

$$\tilde{\mathbf{K}}_{0e}(\omega) = EI_0 \int_{x=0}^L \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\} \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\}^T dx \quad (2.65)$$

$$= \frac{EI_0 b^3}{2} \begin{bmatrix} bL - cs & 1 - c^2 & cS - sC & -1 + cC - sS \\ 1 - c^2 & cs + bL & 1 - cC - sS & -cS - sC \\ cS - sC & 1 - cC - sS & CS - bL & -1 + C^2 \\ -1 + cC - sS & -cS - sC & -1 + C^2 & CS + bL \end{bmatrix} \quad (2.66)$$

The deterministic part of the stiffness matrix can be obtained from Eq. (2.21) using the  $\mathbf{\Gamma}(\omega)$  matrix defined in Eq. (2.63). The term  $\tilde{\mathbf{M}}_{0e}(\omega)$  can be obtained in a similar way as

$$\tilde{\mathbf{M}}_{0e}(\omega) = m_0 \int_{x=0}^L \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \quad (2.67)$$

$$= \frac{m_0}{2b} \begin{bmatrix} bL - cs & 1 - c^2 & -cS + sC & 1 - cC + sS \\ 1 - c^2 & cs + bL & -1 + cC + sS & cS + sC \\ -cS + sC & -1 + cC + sS & CS - bL & -1 + C^2 \\ 1 - cC + sS & cS + sC & -1 + C^2 & CS + bL \end{bmatrix} \quad (2.68)$$

The deterministic mass matrix can be obtained from the above equation as  $\mathbf{M}_{0e}(\omega) = \mathbf{\Gamma}(\omega) \tilde{\mathbf{M}}_{0e}(\omega) \mathbf{\Gamma}^T(\omega)$ .

To obtain the matrices associated with the random components, note that for each  $j$  there will be two different matrices corresponding to the two eigenfunctions defined in equations (2.5) and (2.6). Like the axial vibration of rods, the element stiffness matrix can be expressed as Eq. (2.42) where The matrix  $\tilde{\Delta \mathbf{K}}_e(\omega)$  can be expanded utilizing the Karhunen-Loève expansion as Eq. (2.43).

The matrices  $\tilde{\mathbf{K}}_{je}(\omega)$  can be obtained using the integrals of the form Eq. (2.24). Using the expression of the eigenfunction for the odd values of  $j$  as in Eq. (2.5) one has

$$\begin{aligned} \tilde{\mathbf{K}}_{je}(\omega) &= \int_0^L \frac{\epsilon_{EI} EI_0 \cos(\alpha_j(-L/2 + x))}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\} \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\}^T dx \quad (2.69) \\ &= \frac{\epsilon_{EI} EI_0}{\sqrt{(L/2 + c_\alpha s_\alpha / \alpha_j)}} \hat{\mathbf{K}}_j \end{aligned}$$

where  $c_\alpha, s_\alpha$  are defined in Eq. (2.46) and  $\hat{\mathbf{K}}_j \in \mathbb{C}^{4 \times 4}$  is a symmetric matrix obtained in Appendix A. In a similar manner, using the expression of the eigenfunction for the even values of  $j$  as in Eq. (2.6) one has

$$\tilde{\mathbf{K}}_{je}(\omega) = \int_0^L \frac{\epsilon_{EI} EI_0 \sin(\alpha_j(-L/2 + x))}{\sqrt{L/2 - \frac{\sin(\alpha_j L)}{2\alpha_j}}} \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\} \left\{ \frac{\partial^2 \mathbf{s}(x, \omega)}{\partial x^2} \right\}^T dx \quad (2.70)$$

$$= \frac{\epsilon_{EI}EI_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \widehat{\mathbf{K}}_j$$

The mass matrix can also be represented as above. The eigenvalues and eigenfunctions corresponding to the random field  $H_m(x, \theta)$  needs to be used to obtain the elements of  $\widetilde{\mathbf{M}}_{je}(\omega)$ . Using the expression of the eigenfunction for the odd values of  $j$  as in Eq. (2.5) one has

$$\begin{aligned} \widetilde{\mathbf{M}}_{je}(\omega) &= \int_0^L \frac{\epsilon_m m_0 \cos(\alpha_j(-L/2 + x))}{\sqrt{L/2 + \frac{\sin(\alpha_j L)}{2\alpha_j}}} \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \\ &= \frac{\epsilon_m m_0}{\sqrt{(L/2 + c_\alpha s_\alpha/\alpha_j)}} \widehat{\mathbf{M}}_j \end{aligned} \quad (2.71)$$

In the above expression the eigenvalues  $\alpha_j$  should be obtained by solving the transcendental Eq. (2.5). In a similar manner, using the expression of the eigenfunction for the even values of  $j$  as in Eq. (2.6) one has

$$\begin{aligned} \widetilde{\mathbf{M}}_{je}(\omega) &= \int_0^L \frac{\epsilon_m m_0 \sin(\alpha_j(-L/2 + x))}{\sqrt{L/2 - \frac{\sin(2\alpha_j a)}{2\alpha_j}}} \mathbf{s}(x, \omega) \mathbf{s}^T(x, \omega) dx \\ &= \frac{\epsilon_m m_0}{\sqrt{(L/2 - c_\alpha s_\alpha/\alpha_j)}} \widehat{\mathbf{M}}_j \end{aligned} \quad (2.72)$$

Equations (2.69)–(2.72) completely define the random parts of the element stiffness and mass matrices. The definite integrals appearing in these expressions can be evaluated in closed-form. This further reduces the computational cost in deriving the element matrices. The exact closed-form expression of the elements of the above four matrices are given in Appendix A.

### 2.6.3 Numerical Illustrations

A simple numerical example is considered to illustrate the application of the matrices derived for the Euler-Bernoulli beam. The mean material properties are considered as  $\rho_0 = 7800 \text{kg/m}^3$  and  $E_0 = 210 \text{GPa}$ , values corresponding to steel. The length of the beam is  $L = 1.5 \text{m}$  and the rectangular cross section has width  $40.06 \text{mm}$  and thickness  $2.05 \text{mm}$ . The area moment of inertia of the cross-section  $I = 2.876 \times 10^{-11} \text{m}^4$ . A clamped-free boundary condition is considered for this example. Using these values we have  $EI_0 = 5.752 \text{Nm}^2$  and  $m_0 = \rho_0 A_0 = 0.6406 \text{kg/m}$ . The standard deviations of both the random fields are assumed to be 10% of their mean values, that is,  $\epsilon_{EI} = 0.1EI_0$  and  $\epsilon_m = 0.1m_0$ . The damping coefficients are assumed to be  $c_1 = 6.15 \times 10^{-5}EI_0$  and  $c_2 = 0.09m_0$ . The correlation length of the random fields describing  $EI(x)$  and  $m(x)$  is assumed to be  $L/2$ . We consider the displacement response at the free end of the beam due to an unit harmonic vertical force at that end. The response is calculated upto 200Hz covering the first ten vibration modes of the system. The response of the deterministic system, the mean and the standard deviation

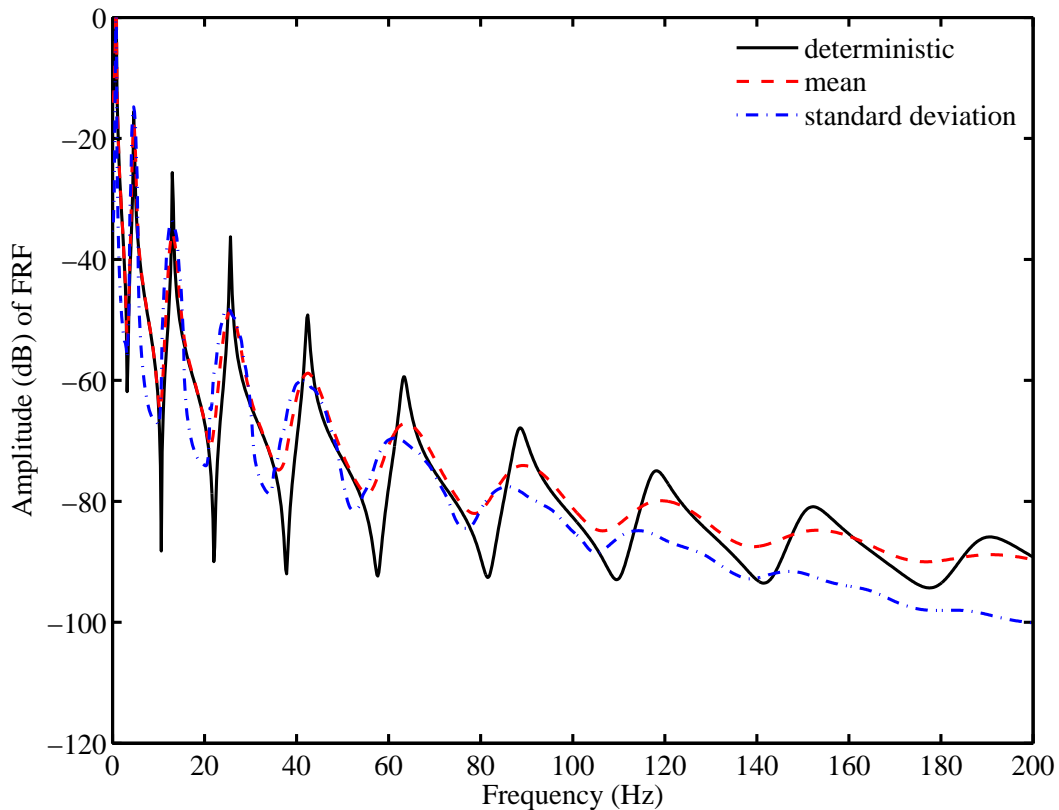


Figure 2.6: The amplitude of the displacement frequency response function at the free end of the damped beam with random properties.

of the absolute value of the response are shown in Figure 2.6. These results are obtained using Monte Carlo simulation with 4000 samples. In total 18 terms are used for the KL expansion. With this number of terms, the last eigenvalue of the KL expansion becomes less than 5% of the first eigenvalue. The element matrices associated with 18 random variables are obtained using the closed-form expression derived the previous section. The phase of the frequency response function at the free end of the beam is shown in Figure 2.7. The phase do not change sign because we are considering the driving point response. In both Figs. 2.6 and 2.7 the mean curve is different from the deterministic curve. This difference is larger at higher frequencies. At lower frequencies, the standard deviation is biased by the mean. But as we approach the higher frequencies, the standard deviation curve flattens. These results are obtained using a single spectral element although ten modes of vibration exist within the frequency range considered.

## 2.7 Conclusions

The basic formulation for Doubly Spectral Stochastic Finite Element Method (DSSFEM) for damped linear dynamical systems with distributed parametric uncertainty has been derived. This new approach simultaneously utilizes the spectral representations in the frequency and

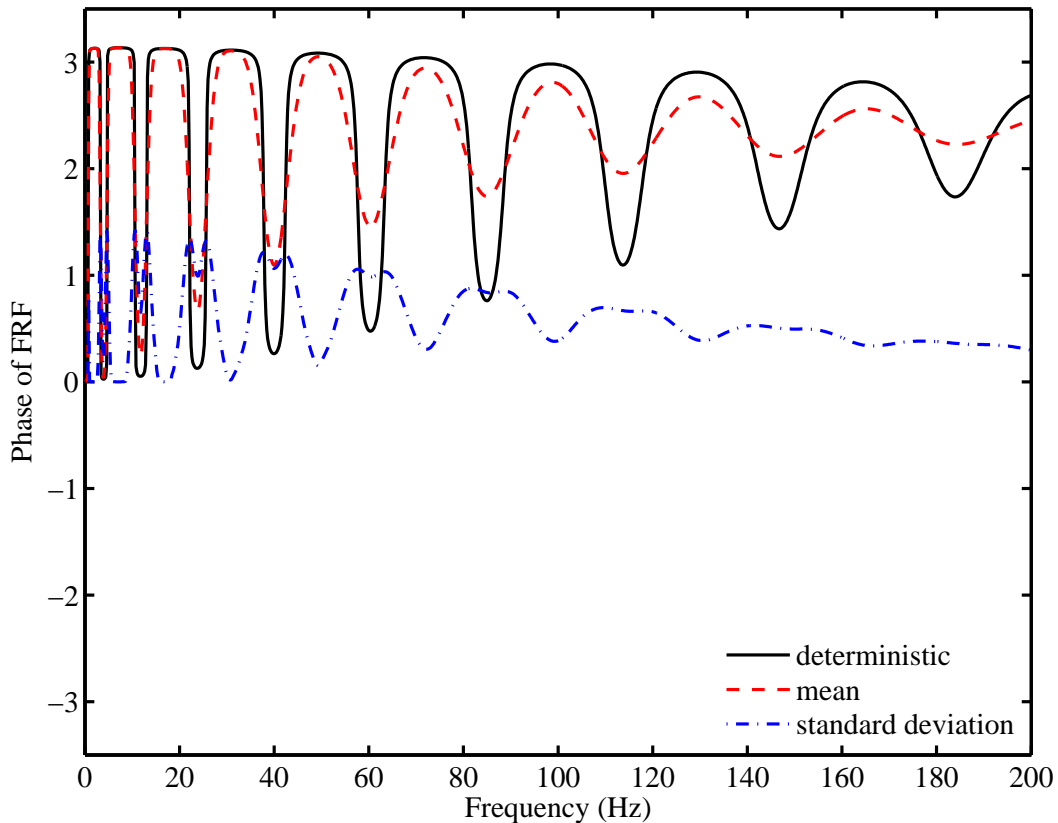


Figure 2.7: The phase of the displacement frequency response function at the free end of the damped beam with random properties.

random domains. The spatial displacement fields are discretized using frequency-adaptive complex shape functions while the spatial random fields are discretized using the Karhunen-Loève expansion. The frequency-adaptive shape functions are obtained from the spectral analysis of the underlying deterministic system, while the Karhunen-Loève expansion is obtained from the spectral decomposition of the autocorrelation function of the spatial random field. In spite of the fact that these two spectral approaches existed for well over three decades, there has not been much overlap between them in literature. In this chapter these two spectral techniques have been unified with the aim that the unified approach would outperform any of the spectral methods considered on its own. The resulting frequency dependent random element matrices in general turn out to be complex symmetric matrices. The main computational advantage of the proposed approach is that the fine spatial discretisation will not be necessary for high and mid-frequency vibration analysis. This in turn results in smaller size matrices to be inverted. The detailed derivations for rods in axial vibration and beams in bending vibration are given. Closed-form expressions of the element stiffness and mass matrices have been derived for the stochastic parametric fields with exponential autocorrelation function. Numerical examples have been given to illustrate the applicability of the proposed method.

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The calculation of dynamic response using the DSSFEM requires the inversion of a complex random symmetric matrix for every frequency. A limitation of the proposed method is that Monte Carlo simulation is necessary for this step. Although the matrix sizes are smaller using the DSSFEM compared to conventional SFEM, this step still requires considerable computational effort. Further research is necessary to develop analytical methods in this direction. Further research is also necessary to extract eigenvalues from the complex random matrices obtained using the proposed method.

# Appendix A

## Expression of spectral element matrices associated with the KL expansion for the bending vibration of beam

This appendix gives the explicit expressions for the spectral stiffness and mass matrices associated with the KL expansion for the bending vibration of beam. The elements of the stiffness matrix associated with the odd values of  $j$  in Eq. (2.69) can be obtained as

$$\begin{aligned}
\widehat{K}_{11} &= \frac{4b^6 s_{\alpha_j} - 2b^5 \alpha_j c_{\alpha_j} c s + (-\alpha_j^2 + \alpha_j^2 c^2) s_{\alpha_j} b^4}{-\alpha_j^3 + 4\alpha_j b^2} \\
\widehat{K}_{12} &= \frac{(2-2c^2) c_{\alpha_j} b^5 - b^4 \alpha_j s_{\alpha_j} c s}{-\alpha_j^2 + 4b^2} \\
\widehat{K}_{13} &= \frac{(2cS - 2sC) c_{\alpha_j} b^7 + (2\alpha_j + 2\alpha_j C c) s_{\alpha_j} b^6 + (-\alpha_j^2 C s - \alpha_j^2 c S) c_{\alpha_j} b^5 - b^4 \alpha_j^3 s_{\alpha_j} S s}{4b^4 + \alpha_j^4} \\
\widehat{K}_{14} &= \frac{(-2Ss - 2 + 2Cc) c_{\alpha_j} b^7 + 2b^6 \alpha_j s_{\alpha_j} c s + (-\alpha_j^2 S s + \alpha_j^2 - \alpha_j^2 C c) c_{\alpha_j} b^5 - b^4 \alpha_j^3 C s_{\alpha_j} s}{4b^4 + \alpha_j^4} \\
\widehat{K}_{22} &= \frac{4b^6 s_{\alpha_j} + 2b^5 \alpha_j c_{\alpha_j} c s + (-\alpha_j^2 - \alpha_j^2 c^2) s_{\alpha_j} b^4}{-\alpha_j^3 + 4\alpha_j b^2} \\
\widehat{K}_{23} &= \frac{(2-2Ss - 2Cc) c_{\alpha_j} b^7 - 2b^6 \alpha_j C s_{\alpha_j} s + (-\alpha_j^2 C c + \alpha_j^2 S s + \alpha_j^2) c_{\alpha_j} b^5 - b^4 \alpha_j^3 s_{\alpha_j} c s}{4b^4 + \alpha_j^4} \\
\widehat{K}_{24} &= \frac{(-2cS - 2sC) c_{\alpha_j} b^7 - 2b^6 \alpha_j s_{\alpha_j} S s + (\alpha_j^2 C s - \alpha_j^2 c S) c_{\alpha_j} b^5 + (-\alpha_j^3 - \alpha_j^3 C c) s_{\alpha_j} b^4}{4b^4 + \alpha_j^4} \\
\widehat{K}_{33} &= \frac{-4b^6 s_{\alpha_j} + 2b^5 \alpha_j S C c_{\alpha_j} + (\alpha_j^2 C^2 - \alpha_j^2) s_{\alpha_j} b^4}{4\alpha_j b^2 + \alpha_j^3} \\
\widehat{K}_{34} &= \frac{(-2 + 2C^2) c_{\alpha_j} b^5 + b^4 \alpha_j S C s_{\alpha_j}}{4b^2 + \alpha_j^2} \\
\widehat{K}_{44} &= \frac{4b^6 s_{\alpha_j} + 2b^5 \alpha_j S C c_{\alpha_j} + (\alpha_j^2 C^2 + \alpha_j^2) s_{\alpha_j} b^4}{4\alpha_j b^2 + \alpha_j^3}.
\end{aligned}$$

The subscript  $j$  is omitted in  $\widehat{K}$  for notational convenience. Because the matrix is symmetric, only the upper triangular part is shown. All the terms appearing in the above expressions have been defined in the main body of the chapter. The elements of the stiffness matrix associated with the even values of  $j$  in Eq. (2.70) can be obtained as

$$\begin{aligned}
\widehat{K}_{11} &= \frac{-2b^5 s_{\alpha_j} c s + (\alpha_j - \alpha_j c^2) c_{\alpha_j} b^4}{-\alpha_j^2 + 4b^2} \\
\widehat{K}_{12} &= \frac{b^4 c_{\alpha_j} c_{\alpha_j} s - 2b^5 c^2 s_{\alpha_j}}{-\alpha_j^2 + 4b^2}
\end{aligned}$$

$$\begin{aligned}
\widehat{K}_{13} &= \frac{(2cS-2sC)s_{\alpha_j}b^7+(-2\alpha_jCc+2\alpha_j)c_{\alpha_j}b^6+(-\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}Ss}{4b^4+\alpha_j^4} \\
\widehat{K}_{14} &= \frac{(2Cc-2Ss+2)s_{\alpha_j}b^7-2b^6\alpha_jc_{\alpha_j}cS+(-\alpha_j^2Cc-\alpha_j^2-\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3Cc_{\alpha_j}s}{4b^4+\alpha_j^4} \\
\widehat{K}_{22} &= \frac{(-2s_{\alpha_j}\alpha_j\cos(\alpha_jL)cs+2c_{\alpha_j}\alpha_j\sin(\alpha_jL)cs)b^5+\left((- \alpha_j^2+\alpha_j^2\cos(\alpha_jL)c^2\right)c_{\alpha_j}+s_{\alpha_j}\alpha_j^2\sin(\alpha_jL)c^2\right)b^4}{-\alpha_j^3+4\alpha_jb^2} \\
\widehat{K}_{23} &= \frac{(-2-2Cc-2Ss)s_{\alpha_j}b^7+2b^6\alpha_jCc_{\alpha_j}s+(-\alpha_j^2Cc-\alpha_j^2+\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}cS}{4b^4+\alpha_j^4} \\
\widehat{K}_{24} &= \frac{(-2cS-2sC)s_{\alpha_j}b^7+2b^6\alpha_jc_{\alpha_j}Ss+(\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+(-\alpha_j^3+\alpha_j^3Cc)c_{\alpha_j}b^4}{4b^4+\alpha_j^4} \\
\widehat{K}_{33} &= \frac{2b^5Scs_{\alpha_j}+(\alpha_j-\alpha_jC^2)c_{\alpha_j}b^4}{4b^2+\alpha_j^2} \\
\widehat{K}_{34} &= \frac{2C^2b^5s_{\alpha_j}-Cb^4\alpha_jSc_{\alpha_j}}{4b^2+\alpha_j^2} \\
\widehat{K}_{44} &= \frac{2b^5Scs_{\alpha_j}+(\alpha_j-\alpha_jC^2)c_{\alpha_j}b^4}{4b^2+\alpha_j^2}.
\end{aligned}$$

The elements of the mass matrix associated with the odd values of  $j$  in Eq. (2.71) can be obtained as

$$\begin{aligned}
\widehat{M}_{11} &= \frac{-2b^5s_{\alpha_j}cs+(\alpha_j-\alpha_jc^2)c_{\alpha_j}b^4}{-\alpha_j^2+4b^2} \\
\widehat{M}_{12} &= \frac{b^4c_{\alpha_j}c_{\alpha_j}s-2b^5c^2s_{\alpha_j}}{-\alpha_j^2+4b^2} \\
\widehat{M}_{13} &= \frac{(2cS-2sC)s_{\alpha_j}b^7+(-2\alpha_jCc+2\alpha_j)c_{\alpha_j}b^6+(-\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}Ss}{4b^4+\alpha_j^4} \\
\widehat{M}_{14} &= \frac{(2Cc-2Ss+2)s_{\alpha_j}b^7-2b^6\alpha_jc_{\alpha_j}cS+(-\alpha_j^2Cc-\alpha_j^2-\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3Cc_{\alpha_j}s}{4b^4+\alpha_j^4} \\
\widehat{M}_{22} &= \frac{(-2s_{\alpha_j}\alpha_j\cos(\alpha_jL)cs+2c_{\alpha_j}\alpha_j\sin(\alpha_jL)cs)b^5+\left((- \alpha_j^2+\alpha_j^2\cos(\alpha_jL)c^2\right)c_{\alpha_j}+s_{\alpha_j}\alpha_j^2\sin(\alpha_jL)c^2\right)b^4}{-\alpha_j^3+4\alpha_jb^2} \\
\widehat{M}_{23} &= \frac{(-2-2Cc-2Ss)s_{\alpha_j}b^7+2b^6\alpha_jCc_{\alpha_j}s+(-\alpha_j^2Cc-\alpha_j^2+\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}cS}{4b^4+\alpha_j^4} \\
\widehat{M}_{24} &= \frac{(-2cS-2sC)s_{\alpha_j}b^7+2b^6\alpha_jc_{\alpha_j}Ss+(\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+(-\alpha_j^3+\alpha_j^3Cc)c_{\alpha_j}b^4}{4b^4+\alpha_j^4} \\
\widehat{M}_{33} &= \frac{2b^5Scs_{\alpha_j}+(\alpha_j-\alpha_jC^2)c_{\alpha_j}b^4}{4b^2+\alpha_j^2} \\
\widehat{M}_{34} &= \frac{2C^2b^5s_{\alpha_j}-Cb^4\alpha_jSc_{\alpha_j}}{4b^2+\alpha_j^2} \\
\widehat{M}_{44} &= \frac{2b^5Scs_{\alpha_j}+(\alpha_j-\alpha_jC^2)c_{\alpha_j}b^4}{4b^2+\alpha_j^2}.
\end{aligned}$$

The elements of the mass matrix associated with the even values of  $j$  in Eq. (2.72) can be obtained as

$$\begin{aligned}
\widehat{M}_{11} &= \frac{-2b^5s_{\alpha_j}cs+(\alpha_j-\alpha_jc^2)c_{\alpha_j}b^4}{-\alpha_j^2+4b^2} \\
\widehat{M}_{12} &= \frac{b^4c_{\alpha_j}c_{\alpha_j}s-2b^5c^2s_{\alpha_j}}{-\alpha_j^2+4b^2} \\
\widehat{M}_{13} &= \frac{(2cS-2sC)s_{\alpha_j}b^7+(-2\alpha_jCc+2\alpha_j)c_{\alpha_j}b^6+(-\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}Ss}{4b^4+\alpha_j^4} \\
\widehat{M}_{14} &= \frac{(2Cc-2Ss+2)s_{\alpha_j}b^7-2b^6\alpha_jc_{\alpha_j}cS+(-\alpha_j^2Cc-\alpha_j^2-\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3Cc_{\alpha_j}s}{4b^4+\alpha_j^4} \\
\widehat{M}_{22} &= \frac{(-2s_{\alpha_j}\alpha_j\cos(\alpha_jL)cs+2c_{\alpha_j}\alpha_j\sin(\alpha_jL)cs)b^5+\left((- \alpha_j^2+\alpha_j^2\cos(\alpha_jL)c^2\right)c_{\alpha_j}+s_{\alpha_j}\alpha_j^2\sin(\alpha_jL)c^2\right)b^4}{-\alpha_j^3+4\alpha_jb^2} \\
\widehat{M}_{23} &= \frac{(-2-2Cc-2Ss)s_{\alpha_j}b^7+2b^6\alpha_jCc_{\alpha_j}s+(-\alpha_j^2Cc-\alpha_j^2+\alpha_j^2Ss)s_{\alpha_j}b^5+b^4\alpha_j^3c_{\alpha_j}cS}{4b^4+\alpha_j^4} \\
\widehat{M}_{24} &= \frac{(-2cS-2sC)s_{\alpha_j}b^7+2b^6\alpha_jc_{\alpha_j}Ss+(\alpha_j^2Cs-\alpha_j^2cS)s_{\alpha_j}b^5+(-\alpha_j^3+\alpha_j^3Cc)c_{\alpha_j}b^4}{4b^4+\alpha_j^4} \\
\widehat{M}_{33} &= \frac{2b^5Scs_{\alpha_j}+(\alpha_j-\alpha_jC^2)c_{\alpha_j}b^4}{4b^2+\alpha_j^2} \\
\widehat{M}_{34} &= \frac{2C^2b^5s_{\alpha_j}-Cb^4\alpha_jSc_{\alpha_j}}{4b^2+\alpha_j^2}
\end{aligned}$$

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$$\widehat{M}_{44} = \frac{2b^5 S C s_{\alpha_j} + (\alpha_j - \alpha_j C^2) c_{\alpha_j} b^4}{4b^2 + \alpha_j^2}.$$



# Appendix B

## Expressions for elements of matrices $\alpha^l(\omega)$ and $X_l$

### B.1 Expressions for elements of matrices $[\alpha^l(\omega)]$

The matrices  $[\alpha^l(\omega)]$  can be written as

$$[\alpha^l(\omega)] = \begin{bmatrix} [\beta^l(\omega)]_{(4 \times 4)} & \mathbf{O}_{(4 \times 2)} \\ \mathbf{O}_{(2 \times 4)} & \mathbf{O}_{(2 \times 2)} \end{bmatrix}_{(6 \times 6)} \quad \text{for } l = 1..10$$

$$[\alpha^l(\omega)] = \begin{bmatrix} \mathbf{O}_{(4 \times 4)} & \mathbf{O}_{(4 \times 2)} \\ \mathbf{O}_{(2 \times 4)} & [\theta^l(\omega)]_{(2 \times 2)} \end{bmatrix}_{(6 \times 6)} \quad \text{for } l = 11..13 \quad (\text{B.1})$$

where  $\mathbf{O}$  is the null matrix.

In the following expressions  $C = \cosh bl$ ,  $c = \cos bl$ ,  $S = \sinh bl$ ,  $s = \sin bl$  and

$$b^4 = \frac{m_0 \omega^2}{EI_0} \quad (\text{B.2})$$

$$\begin{aligned} \beta_{1,1}^1 &= \frac{1}{4} \frac{(cS+C s)^2}{(-1+cC)^2} & \beta_{1,2}^1 &= \frac{1}{4} \frac{(cS+C s)(-1+cC+sS)}{(-1+cC)^2 b} & \beta_{1,3}^1 &= -\frac{1}{4} \frac{(cS+C s)(S+s)}{(-1+cC)^2} \\ \beta_{1,4}^1 &= -\frac{1}{4} \frac{(cS+C s)(-C+c)}{(-1+cC)^2 b} & \beta_{2,2}^1 &= \frac{1}{4} \frac{(-1+cC+sS)^2}{b^2 (-1+cC)^2} & \beta_{2,3}^1 &= -\frac{1}{4} \frac{(-1+cC+sS)(S+s)}{b(-1+cC)^2} \\ \beta_{2,4}^1 &= -\frac{1}{4} \frac{(-1+cC+sS)(-C+c)}{b^2 (-1+cC)^2} & \beta_{3,3}^1 &= \frac{1}{4} \frac{(S+s)^2}{(-1+cC)^2} & \beta_{3,4}^1 &= \frac{1}{4} \frac{(S+s)(-C+c)}{(-1+cC)^2 b} \\ \beta_{4,4}^1 &= \frac{1}{4} \frac{(-C+c)^2}{b^2 (-1+cC)^2} \end{aligned}$$

$$\begin{aligned} \beta_{1,1}^2 &= \frac{1}{2} \frac{(cS+C s)(-sS-1+cC)}{(-1+cC)^2} & \beta_{1,2}^2 &= \frac{1}{4} \frac{c^2 S^2 - C^2 s^2 - s^2 S^2 + 1 - 2cC + c^2 C^2}{(-1+cC)^2 b} \\ \beta_{1,3}^2 &= -\frac{1}{4} \frac{c^2 S - C^2 s + 2C s c - s S^2 - s^2 S - S - s}{(-1+cC)^2} \\ \beta_{1,4}^2 &= -\frac{1}{4} \frac{c S^2 - 2c S s + 2C s S - C s^2 + C - c - c C^2 + c^2 C}{(-1+cC)^2 b} \\ \beta_{2,2}^2 &= \frac{1}{2} \frac{(-1+cC+sS)(-C+c)}{b^2 (-1+cC)^2} & \beta_{2,3}^2 &= -\frac{1}{4} \frac{C - c - c C^2 + c^2 C - 2C s S + 2c S s - C s^2 + c S^2}{(-1+cC)^2 b} \\ \beta_{2,4}^2 &= -\frac{1}{4} \frac{-S + s - 2C s c + s S^2 - s^2 S + C^2 s + c^2 S}{b^2 (-1+cC)^2} & \beta_{3,3}^2 &= \frac{1}{2} \frac{(S+s)(-C+c)}{(-1+cC)^2} \\ \beta_{3,4}^2 &= \frac{1}{4} \frac{S^2 - s^2 + C^2 - 2cC + c^2}{(-1+cC)^2 b} & \beta_{4,4}^2 &= \frac{1}{2} \frac{(-C+c)(S-s)}{b^2 (-1+cC)^2} \end{aligned}$$

$$\begin{aligned}\beta_{1,1}^3 &= -\frac{1}{2} \frac{(cS+C s)^2}{(-1+cC)^2} & \beta_{1,2}^3 &= -\frac{1}{2} \frac{(cS+C s)sS}{(-1+cC)^2 b} & \beta_{1,3}^3 &= \frac{1}{2} \frac{(cS+C s)(S+s)}{(-1+cC)^2} \\ \beta_{1,4}^3 &= \frac{1}{2} \frac{(cS+C s)(-C+c)}{(-1+cC)^2 b} & \beta_{2,2}^3 &= \frac{1}{2} \frac{(-sS-1+cC)(-1+cC+sS)}{b^2(-1+cC)^2} & \beta_{2,3}^3 &= \frac{1}{2} \frac{(S+s)sS}{(-1+cC)^2 b} \\ \beta_{2,4}^3 &= \frac{1}{2} \frac{(-C+c)sS}{b^2(-1+cC)^2} & \beta_{3,3}^3 &= -\frac{1}{2} \frac{(S+s)^2}{(-1+cC)^2} & \beta_{3,4}^3 &= -\frac{1}{2} \frac{(S+s)(-C+c)}{(-1+cC)^2 b} & \beta_{4,4}^3 &= -\frac{1}{2} \frac{(-C+c)^2}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}\beta_{1,1}^4 &= \frac{1}{2} \frac{(cS+C s)(-1+cC+sS)}{(-1+cC)^2} & \beta_{1,2}^4 &= -\frac{1}{4} \frac{c^2 S^2 - C^2 s^2 - 1 + 2cC + 2sS - c^2 C^2 - 2cSCs - s^2 S^2}{(-1+cC)^2 b} \\ \beta_{1,3}^4 &= \frac{1}{4} \frac{-2cSC + c^2 S - C^2 s + S + s - sS^2 - s^2 S}{(-1+cC)^2} \\ \beta_{1,4}^4 &= -\frac{1}{4} \frac{-cS^2 + 2cSs - 2CsS + C s^2 + C - c - cC^2 + c^2 C}{(-1+cC)^2 b} \\ \beta_{2,2}^4 &= -\frac{1}{2} \frac{(-1+cC+sS)(-Cs+cS)}{b^2(-1+cC)^2} & \beta_{2,3}^4 &= \frac{1}{4} \frac{C - c - cC^2 + c^2 C - 2CsS + 2cSs - C s^2 + cS^2}{(-1+cC)^2 b} \\ \beta_{2,4}^4 &= \frac{1}{4} \frac{-S + s - 2Cs + cS^2 - s^2 S + C^2 s + c^2 S}{b^2(-1+cC)^2} & \beta_{3,3}^4 &= -\frac{1}{2} \frac{(S+s)(-C+c)}{(-1+cC)^2} \\ \beta_{3,4}^4 &= -\frac{1}{4} \frac{S^2 - s^2 + C^2 - 2cC + c^2}{(-1+cC)^2 b} & \beta_{4,4}^4 &= -\frac{1}{2} \frac{(-C+c)(S-s)}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}\beta_{1,1}^5 &= \frac{1}{4} \frac{(-sS-1+cC)^2}{(-1+cC)^2} & \beta_{1,2}^5 &= \frac{1}{4} \frac{(-sS-1+cC)(-Cs+cS)}{(-1+cC)^2 b} & \beta_{1,3}^5 &= -\frac{1}{4} \frac{(-sS-1+cC)(-C+c)}{(-1+cC)^2} \\ \beta_{1,4}^5 &= -\frac{1}{4} \frac{(-sS-1+cC)(S-s)}{(-1+cC)^2 b} & \beta_{2,2}^5 &= \frac{1}{4} \frac{(-Cs+cS)^2}{b^2(-1+cC)^2} \\ \beta_{2,3}^5 &= -\frac{1}{4} \frac{(-Cs+cS)(-C+c)}{(-1+cC)^2 b} & \beta_{2,4}^5 &= -\frac{1}{4} \frac{(-Cs+cS)(S-s)}{b^2(-1+cC)^2} \\ \beta_{3,3}^5 &= \frac{1}{4} \frac{(-C+c)^2}{(-1+cC)^2} & \beta_{3,4}^5 &= \frac{1}{4} \frac{(-C+c)(S-s)}{(-1+cC)^2 b} & \beta_{4,4}^5 &= \frac{1}{4} \frac{(S-s)^2}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}\beta_{1,1}^6 &= -\frac{1}{2} \frac{(cS+C s)(-sS-1+cC)}{(-1+cC)^2} \\ \beta_{1,2}^6 &= -\frac{1}{4} \frac{(-s^2 S^2 - 2sS + 2cSCs - 1 + 2cC - c^2 C^2 + c^2 S^2 - C^2 s^2)}{(-1+cC)^2 b} \\ \beta_{1,3}^6 &= \frac{1}{4} \frac{c^2 S - C^2 s + 2Cs - c - sS^2 - s^2 S - S - s}{(-1+cC)^2} & \beta_{1,4}^6 &= \frac{1}{4} \frac{cS^2 - 2cSs + 2CsS - C s^2 + C - c - cC^2 + c^2 C}{(-1+cC)^2 b} \\ \beta_{2,2}^6 &= \frac{1}{2} \frac{(-sS-1+cC)(-Cs+cS)}{b^2(-1+cC)^2} & \beta_{2,3}^6 &= -\frac{1}{4} \frac{2CsS + C s^2 - cS^2 - 2cSs + C - c - cC^2 + c^2 C}{(-1+cC)^2 b} \\ \beta_{2,4}^6 &= \frac{1}{4} \frac{C^2 s - 2cSC + c^2 S + sS^2 - s^2 S + S - s}{b^2(-1+cC)^2} & \beta_{3,3}^6 &= -\frac{1}{2} \frac{(S+s)(-C+c)}{(-1+cC)^2} \\ \beta_{3,4}^6 &= -\frac{1}{4} \frac{S^2 - s^2 + C^2 - 2cC + c^2}{(-1+cC)^2 b} & \beta_{4,4}^6 &= -\frac{1}{2} \frac{(-C+c)(S-s)}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}\beta_{1,1}^7 &= \frac{1}{2} \frac{(-sS-1+cC)(-1+cC+sS)}{(-1+cC)^2} & \beta_{1,2}^7 &= \frac{1}{2} \frac{(-Cs+cS)sS}{(-1+cC)^2 b} \\ \beta_{1,3}^7 &= -\frac{1}{2} \frac{(-C+c)sS}{(-1+cC)^2} & \beta_{1,4}^7 &= -\frac{1}{2} \frac{(S-s)sS}{(-1+cC)^2 b} \\ \beta_{2,2}^7 &= -\frac{1}{2} \frac{(-Cs+cS)^2}{b^2(-1+cC)^2} & \beta_{2,3}^7 &= \frac{1}{2} \frac{(-Cs+cS)(-C+c)}{(-1+cC)^2 b} \\ \beta_{2,4}^7 &= \frac{1}{2} \frac{(-Cs+cS)(S-s)}{b^2(-1+cC)^2} & \beta_{3,3}^7 &= -\frac{1}{2} \frac{(-C+c)^2}{(-1+cC)^2} & \beta_{3,4}^7 &= -\frac{1}{2} \frac{(-C+c)(S-s)}{(-1+cC)^2 b} \\ \beta_{4,4}^7 &= -\frac{1}{2} \frac{(S-s)^2}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}\beta_{1,1}^8 &= \frac{1}{4} \frac{(cS+C s)^2}{(-1+cC)^2} & \beta_{1,2}^8 &= -\frac{1}{4} \frac{(cS+C s)(-sS-1+cC)}{(-1+cC)^2 b} & \beta_{1,3}^8 &= -\frac{1}{4} \frac{(cS+C s)(S+s)}{(-1+cC)^2} \\ \beta_{1,4}^8 &= -\frac{1}{4} \frac{(cS+C s)(-C+c)}{(-1+cC)^2 b} & \beta_{2,2}^8 &= \frac{1}{4} \frac{(-sS-1+cC)^2}{b^2(-1+cC)^2} & \beta_{2,3}^8 &= \frac{1}{4} \frac{(-sS-1+cC)(S+s)}{(-1+cC)^2 b} \\ \beta_{2,4}^8 &= \frac{1}{4} \frac{(-sS-1+cC)(-C+c)}{b^2(-1+cC)^2} & \beta_{3,3}^8 &= \frac{1}{4} \frac{(S+s)^2}{(-1+cC)^2} & \beta_{3,4}^8 &= \frac{1}{4} \frac{(S+s)(-C+c)}{(-1+cC)^2 b} \\ \beta_{4,4}^8 &= \frac{1}{4} \frac{(-C+c)^2}{b^2(-1+cC)^2}\end{aligned}$$

$$\begin{aligned}
\beta_{1,1}^9 &= -\frac{1}{2} \frac{(cS+C s)(-1+cC+s S)}{(-1+cC)^2} & \beta_{1,2}^9 &= \frac{1}{4} \frac{c^2 S^2 - C^2 s^2 - s^2 S^2 + 1 - 2cC + c^2 C^2}{(-1+cC)^2 b} \\
\beta_{1,3}^9 &= -\frac{1}{4} \frac{-2cS C + c^2 S - C^2 s + S + s - s S^2 - s^2 S}{(-1+cC)^2} \\
\beta_{1,4}^9 &= \frac{1}{4} \frac{-cS^2 + 2cS s - 2C s S + C s^2 + C - c - cC^2 + c^2 C}{(-1+cC)^2 b} & \beta_{2,2}^9 &= -\frac{1}{2} \frac{(-sS - 1 + cC)(-Cs + cS)}{b^2 (-1+cC)^2} \\
\beta_{2,3}^9 &= \frac{1}{4} \frac{2C s S + C s^2 - cS^2 - 2cS s + C - c - cC^2 + c^2 C}{(-1+cC)^2 b} & \beta_{2,4}^9 &= -\frac{1}{4} \frac{C^2 s - 2cS C + c^2 S + s S^2 - s^2 S + S - s}{b^2 (-1+cC)^2} \\
\beta_{3,3}^9 &= \frac{1}{2} \frac{(S+s)(-C+c)}{(-1+cC)^2} & \beta_{3,4}^9 &= \frac{1}{4} \frac{S^2 - s^2 + C^2 - 2cC + c^2}{(-1+cC)^2 b} \\
\beta_{4,4}^9 &= \frac{1}{2} \frac{(-C+c)(S-s)}{b^2 (-1+cC)^2}
\end{aligned}$$

$$\begin{aligned}
\beta_{1,1}^{10} &= \frac{1}{4} \frac{(-1+cC+s S)^2}{(-1+cC)^2} & \beta_{1,2}^{10} &= -\frac{1}{4} \frac{(-1+cC+s S)(-Cs + cS)}{(-1+cC)^2 b} & \beta_{1,3}^{10} &= \frac{1}{4} \frac{(-1+cC+s S)(-C+c)}{(-1+cC)^2} \\
\beta_{1,4}^{10} &= \frac{1}{4} \frac{(-1+cC+s S)(S-s)}{(-1+cC)^2 b} & \beta_{2,2}^{10} &= \frac{1}{4} \frac{(-Cs + cS)^2}{b^2 (-1+cC)^2} & \beta_{2,3}^{10} &= -\frac{1}{4} \frac{(-Cs + cS)(-C+c)}{(-1+cC)^2 b} \\
\beta_{2,4}^{10} &= -\frac{1}{4} \frac{(-Cs + cS)(S-s)}{b^2 (-1+cC)^2} & \beta_{3,3}^{10} &= \frac{1}{4} \frac{(-C+c)^2}{(-1+cC)^2} & \beta_{3,4}^{10} &= \frac{1}{4} \frac{(-C+c)(S-s)}{(-1+cC)^2 b} & \beta_{4,4}^{10} &= \frac{1}{4} \frac{(S-s)^2}{b^2 (-1+cC)^2}
\end{aligned}$$

Expressions for elements of  $\theta^l(\omega)$  are given by

$$\begin{aligned}
\theta_{1,1}^{11} &= \cot(al)^2 & \theta_{1,2}^{11} &= -\cot(al) \csc(al) & \theta_{2,2}^{11} &= \csc(al)^2 \\
\theta_{1,1}^{12} &= -2 \cot(al) & \theta_{1,2}^{12} &= \csc(al) & \theta_{2,2}^{12} &= 0 \\
\theta_{1,1}^{13} &= 1 & \theta_{1,2}^{13} &= 0 & \theta_{2,2}^{13} &= 0
\end{aligned}$$

with

$$a^2 = \frac{m_0 \omega^2}{AE_0} \quad (\text{B.3})$$

It may be noted that the matrices  $[\beta^l(\omega)]$  and  $[\theta^l(\omega)]$  are all symmetric and only the upper triangle part of these matrices have been listed above.

## B.2 Expressions of the dynamic weighted integrals $X_l$

Expression of  $X_l$  for  $l = 1..10$  depends upon the sign of  $b^4$  and are shown to be given by

**Case 1 :  $b^4 > 0$**

$$X_1 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sin(bx)^2 + EI_0 \epsilon_3 f_3(x) \sin(bx)^2 b^4 \} dx$$

$$X_2 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sin(bx) \cos(bx) + EI_0 \epsilon_3 f_3(x) \sin(bx) b^4 \cos(bx) \} dx$$

$$X_3 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sin(bx) \sinh(bx) - EI_0 \epsilon_3 f_3(x) \sin(bx) b^4 \sinh(bx) \} dx$$

$$X_4 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sin(bx) \cosh(bx) - EI_0 \epsilon_3 f_3(x) \sin(bx) b^4 \cosh(bx) \} dx$$

$$X_5 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \cos(bx)^2 + EI_0 \epsilon_3 f_3(x) \cos(bx)^2 b^4 \} dx$$

$$X_6 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \cos(bx) \sinh(bx) - EI_0 \epsilon_3 f_3(x) \cos(bx) b^4 \sinh(bx) \} dx$$

$$X_7 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \cos(bx) \cosh(bx) - EI_0 \epsilon_3 f_3(x) \cos(bx) b^4 \cosh(bx) \} dx$$

$$X_8 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sinh(bx)^2 + EI_0 \epsilon_3 f_3(x) \sinh(bx)^2 b^4 \} dx$$

$$X_9 = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \sinh(bx) \cosh(bx) + EI_0 \epsilon_3 f_3(x) \sinh(bx) b^4 \cosh(bx) \} dx$$

$$X_{10} = \int_0^L \{ (k_0 \epsilon_1 f_1(x) - m_0 \omega^2 \epsilon_2 f_2(x)) \cosh(bx)^2 + EI_0 \epsilon_3 f_3(x) \cosh(bx)^2 b^4 \} dx$$

for  $l = 11..13$

$$X_{11} = \int_0^L \{ -m_0 \omega^2 \epsilon_2 f_2(x) \sin(ax)^2 + AE_0 \epsilon_4 f_4(x) \cos(ax)^2 a^2 \} dx$$

$$X_{12} = \int_0^L \{ -m_0 \omega^2 \epsilon_2 f_2(x) \sin(ax) \cos(ax) - AE_0 \epsilon_4 f_4(x) \cos(ax) a^2 \sin(ax) \} dx$$

$$X_{13} = \int_0^L \{ -m_0 \omega^2 \epsilon_2 f_2(x) \cos(ax)^2 + AE_0 \epsilon_4 f_4(x) \sin(ax)^2 a^2 \} dx$$

for the above expressions  $b$ ,  $b'$  and  $a$  is same as defined in equations (B.2), (??) and (B.3).

# Appendix C

## MAPLE V Program BMSYM

```
#####
#PROGRAM FOR CALCULATING THE SYMBOLIC EXPRESSION OF MEAN OF DYNAMIC
#STIFFNESS MATRIX, MATRIX ASSOCIATED WITH THE WEIGHTED INTEGRALS AND
#EXPRESSIONS OF THE WEIGHTED INTEGRALS IN FLEXURAL MOTION OF THE BEAM
#ELEMENT.
#-----

with(combinat):
with(linalg):

#DEFINING DIFFERENT ARRAYS:
Gamma:=array(1..4,1..4):
N:=array(1..4):
Du:=array(1..4,1..4):
for lr to 10 do
    alpha.lr:=array(1..4,1..4):
od;

s:=array(1..4,[sin(b*x),cos(b*x),sinh(b*x),cosh(b*x)]);
Rb:=array(1..4,1..4,[[[0,1,0,1],[b,0,b,0],
    [sin(b*1),cos(b*1),sinh(b*1),cosh(b*1)],
    [cos(b*1)*b,-sin(b*1)*b,cosh(b*1)*b,sinh(b*1)*b]]]);
t:=inverse(Rb):
Gammas:=map(simplify,transpose(t)):
N:=evalm(Gammas &* s):
nd:=map(diff,N,x$2):
pn:=evalm(N &* transpose(N) ):
pnd:=evalm(nd &* transpose(nd) ):
kr:=evalm( evalm( -b^4* pn) + pnd):
Du1:=evalm(EI*map(simplify,map(normal,
    map(convert,map(int,kr,x=0..1),trig),expanded))):

#CALCULATION OF MATRIX ASSOCIATED WITH WEIGHTED INTEGRALS FOR b^4
#NOT EQUAL TO 0

for i to 4 do
    for j to 4 do
        for k to 4 do
            for r to 4 do
                ll:=(k-1)*4+r-numbcomb(k,2);
```

---

```

        if r < k
            then next
        fi;
        if r = k then
            alphas.ll[i,j]:=Gammas[i,k]*Gammas[j,r]
        else
            alphas.ll[i,j]:=Gammas[i,k]*Gammas[j,r]+ Gammas[i,r]*Gammas[j,k]
        fi;
    od
od;
od;

# EVALUATING THE EXPRESSIONS OF THE WEIGHTED INTEGRALS
for k to 4 do
    for r to 4 do
        ll:=(k-1)*4+r-numbcomb(k,2);
        if r < k
            then next
        fi;
        if r >= k then
            X.ll:= int( (((k_0*epsilon1*f_1(x)
                - m_0*omega^2*epsilon2*f_2(x) )*s[k]*s[r])
                + (EI_0 *epsilon3*f_3(x) * diff(s[k],x$2)
                    * diff(s[r],x$2))),x=0..L)
        fi;
    od
od;

# SIMPLIFYING THE EXPRESSION OF alphas MATRIX TO alpha MATRIX
for ll to 10 do
    for i to 4 do
        for j to 4 do
            alpha.ll[i,j]:=simplify( subs(sin(b*1)=s,cos(b*1)=c,
                sinh(b*1)=S,cosh(b*1)=C,alphas.ll[i,j]) )
        od
    od;
od;

```

---

```
# SIMPLIFYING THE EXPRESSION OF Du1 MATRIX TO Du MATRIX
  for i to 4 do
    for j to 4 do
      Du[i,j]:=simplify( subs(sin(b*1)=s,cos(b*1)=c,
                             sinh(b*1)=S,cosh(b*1)=C,Du1[i,j]) )
    od
  od;

# SIMPLIFYING THE EXPRESSION OF Gammas MATRIX TO Gamma MATRIX
  for i to 4 do
    for j to 4 do
      Gamma[i,j]:= simplify(subs(sin(b*1)=s,cos(b*1)=c,
                             sinh(b*1)=S,cosh(b*1)=C,Gammas[i,j]));
    od
  od;
else fi;
```

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