

# Introduction to the Random Matrix Theory and the Stochastic Finite Element Method<sup>1</sup>



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<sup>1</sup>Part of the presented work is taken from MRes thesis by B Pascual: Random Matrix Approach for Stochastic Finite Element Method, 2008, Swansea University

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# Chapter 1

## Introduction

Uncertainty quantification in a real-life engineering structural system allows a good assessment on the reliability of that system. Uncertainty can be introduced on the one hand by parameters like material properties, geometric parameters, boundary conditions or applied loads, and on the other hand by epistemic uncertainty like lack of knowledge about the model that should represent the system (linear or nonlinear system, viscous or non-viscous damping model, model of structural joints) and numerical methods. Some practical examples of structures where randomness exists, and so they have to be designed with negligible probability of failure, are nuclear power plants, offshore platforms, high-rise buildings or reentry space vehicles, where the system randomness can be introduced by soil variability, earthquake-induced ground motion, random ocean waves, thermal and acoustic loadings and uncertain fatigue effects. The reliability of the design will depend on knowing the effects of those uncertainties in the system response .

Engineering systems are modelled using conservation laws: conservation of momentum, energy and angular momentum, combined with constitutive equations, from where a differential equation of the system will arise. This equation can be solved analytically only for simple geometries or systems: that means generally an analytical solution is not available. Several procedures have been developed to solve the problem using a numerical approach. Finite Element Method (FEM) has become most versatile tool for solving any real-life engineering problem. This method firstly uses Hamilton principle to obtain a weak-form of the differential equation of the system. The geometry of the system is discretized into elements where some shape functions are defined (as in the Galerkin method, but for each element instead of the whole structure). After this discretization process, a matrix equation of the form  $\mathbf{KU} = \mathbf{f}$  (in a static case) needs to be solved, where  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{U}$  the displacement vector (giving the displacement of every node of the structure) and  $\mathbf{f}$  is the force vector.

If a structure is modelled as an interaction between a system operator  $\mathbf{\Lambda}$ , an excitation force  $\mathbf{f}$  ( $\mathbf{\Lambda}\mathbf{U} = \mathbf{f}$ ,  $\mathbf{U}$  being the response), the effect of random excitation in the response of the system has widely been studied elsewhere in the literature (Lin (1967), Yang and

Kapania (1984) and Lin *et al.* (1986)). However, the case when randomness is introduced by the operator is still under study due to its higher difficulty. Particular solutions to some stochastic differential equation (e.g. Itô type (Itô (1951)) and functional integration (Hopf (1952), Lee (1974))), are available for few cases. But problems where stochastic operators's random coefficients possess smooth sample path has no mathematical theory for obtaining the exact solution. The method for finding a solution to the stochastic differential equation needs two steps. The first step consist in modelling adequately the random properties (Benjamin and Cornell (1970) and Vanmarcke (1983)). The second step involves solution of the differential equation.

## 1.1 Random field discretization

In this section some basics of probability theory and random field/process are summarized (Ghanem and Spanos (1991), Sudret and Der-Kiureghian (2000), Papoulis and Pillai (2002) or Keese (2003)). Emphasis has been made on the available discretization schemes of random field/process.

A probability space is denoted by  $\mathcal{L}^2(\Theta, \mathcal{B}, P)$ , where  $\Theta$  is the set of elementary events or sample space, containing all the possible outcomes;  $\mathcal{B}$  is the  $\sigma$ -algebra of events: collection of possible events having well-defined probabilities and  $P$  is the probability measure.

A random variable  $X$  is a function whose domain is  $\Theta$ ;  $X : (\Theta, \mathcal{B}, P) \rightarrow \mathbb{R}$  inducing a probability measure on  $\mathbb{R}$  called the probability distribution of  $X$ ,  $P_X$ . The mathematical expectation, mean variance and  $n$ -th momentum of  $X$  are denoted respectively by  $\mu$ ,  $\sigma^2$  and  $E[X^n]$ . We define  $\{X < k\}$  as the set of experimental outcomes such that after mapping this set with the random variable, the result is less than  $k$ . The cumulative distribution function (CDF)  $F_X$  verifies  $F_X = P\{X < k\} = P_X(-\infty, k)$ , and the probability density function (PDF)  $f_X(x)$  is the derivative of the CDF. The covariance of two independent random variables  $X_1$  and  $X_2$  is

$$\text{Cov}[X_1, X_2] = E[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})] \quad (1.1)$$

Two random variables  $X_1, X_2$  are uncorrelated if  $\text{Cov}[X_1, X_2] = 0$ , and they are orthogonal if  $E[X_1 X_2] = 0$ . A Hilbert space  $\mathbf{H}$  is a generalization of euclidian space ( $\mathbb{R}^3$ ) into  $\mathbb{R}^n$ , is complete and the inner product is defined as:

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx \quad (1.2)$$

The probability space is a Hilbert space, and a random field  $H(x, \theta)$  can be defined as a curve in the probability space  $\mathcal{L}^2$ . That is, a collection of random variables indexed by  $x$  related to the system geometry: for a given  $x_0$ ,  $H(x_0, \theta)$  is a random variable and for a given outcome  $\theta_0$ ,  $H(x, \theta_0)$  is a realization of the field.

The random field is Gaussian if any vector  $\{H(x_1), \dots, H(x_n)\}$  is Gaussian, and is completely defined by its mean  $\mu(X)$ , variance  $\sigma^2(X)$  and autocorrelation coefficient  $\rho(x, x')$ . It is homogenous if its mean and variance are constant and  $\rho$  is a function of the difference  $x' - x$ . A discretization procedure allows to approximate the random field with a finite set of random variables, and the better the approximation is, the less random variables are needed to approximate the random field accurately. There are three groups of discretization schemes, namely, point discretization, average discretization and series expansion. More details are given in Li and Kiureghian (1993a), Matthies *et al.* (1997), Ditlevsen and Madsen (1996) or Sudret and Der-Kiureghian (2000).

### 1.1.1 Point discretization methods

The first group of discretization methods is based on point discretization, the random variables  $\{\chi_i\}$  are selected values of the random field  $H(*)$  at some given points  $x_i$ , and the approximated field  $\hat{H}$  is obtained from  $\chi = \{H(x_1), \dots, H(x_N)\}$ .

- The *shape function method*, introduced by Liu *et al.* (1986a,b) approximates the random field in each element using nodal values  $x_i$  and polynomial shape functions  $N_i$  associated with the element

$$\hat{H} = \sum_{i=1}^q N_i(x)H(x_i) \quad (1.3)$$

with  $x \in \Omega$  where  $\Omega$  is an open set describing the geometry and  $q$  the number of nodes.

- The *midpoint method*, introduced by Der-Kiureghian and Ke (1987), uses as points  $x_i$  the centroid of each element, and then has as many random variables approximating the random field as number of elements; it over-represent the variability of the random field within each element.
- The *integration point method* (Matthies *et al.* (1997)), associate a single random variable to each gauss point of each element appearing in the finite element resolution scheme. This involves that every integration appearing in the Finite Element (FE) model is done by Gauss point method. Results are accurate for short correlation length. However, the total number of random variables involved increases dramatically with the size of the problem.
- The *optimal linear estimation method* (OLE) or *Kringing method*, presented by Li and Kiureghian (1993b), approximates the random field with random variables dependent on nodal values  $\chi = \{H(x_1), \dots, H(x_q)\}$ . The dependence is linear

$$\hat{H}(x) = a(x) + \mathbf{b}_i^T(x)\chi_i \quad (1.4)$$

and coefficients are calculated minimizing the variance of the difference between the approximated and exact random field  $\text{Var} [H(x) - \hat{H}(x)]$  while keeping the mean of that difference equal to zero :  $\text{E} [H(x) - \hat{H}(x)] = 0$ .

### 1.1.2 Average discretization methods

The second group of discretization methods are the *average discretization methods*: the random variables are weighted integrals of the random field over a domain:

$$\chi_i = \int_{\Omega_e} H(x)w(x)d\Omega \quad (1.5)$$

- *Spatial average method* (Vanmarcke and Grigoriu (1983)) uses a random field constant over each element. This constant is the average of the original field over the element, and it has been shown that the variance of the spatial average over an element under-represents the local variance of the random field (Der-Kiureghian and Ke (1987)).
- The *weighted integral method* (Shinozuka and Deodatis (1991), Deodatis (1991, 1990)) considers, for linear elasticity, the element stiffness matrices as random quantities: each random variable is the result of integrating the product of one of the monomials used in the FEM by the random field over each element.

### 1.1.3 Series expansion methods

The last group of discretization methods, *Series expansion methods*, expands any realization of the original random field over a complete set of deterministic functions  $\phi_i$  and truncates the series after a finite number of terms

$$\hat{H}(x, \theta) = \sum_{i=1}^N \chi_i(\theta)\phi_i(x) \quad (1.6)$$

- The most widely used of all series expansion method is *Karhunen-Loève expansion*, which will be discussed little later.
- The *Orthogonal Series Expansion method* (OSE) (Zhang and Ellingwood (1993)) selects a set of deterministic orthogonal functions  $\{h_i(x)\}_{i=1}^{\infty}$  forming a basis for the probability space  $\mathcal{L}^2(\Omega)$ , the coefficients of the expansion are zero mean random variables

$$\chi_i(\theta) = \int_{\Omega} [H(\mathbf{x}, \theta) - \mu(\mathbf{x})]h_i(\mathbf{x})d\Omega \quad (1.7)$$

and the expansion

$$H(\mathbf{x}, \theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \chi_i(\theta)h_i(\mathbf{x}) \quad (1.8)$$

is obtained after computing the covariance matrix  $\Sigma_{\chi\chi}$  of vector  $\boldsymbol{\chi} = \{\chi_1, \dots, \chi_M\}$  where  $M$  is the number of terms of the expansion and  $\mathbf{x}$  is the vector of points  $x_j$ .

- An extension of OSE, using a spectral representation of  $\chi$ , is *Expansion Optimal Linear Estimation method* (EOLE) (Li and Kiureghian (1993a)):

$$\chi(\theta) = \mu_\chi + \sum_{i=1}^N \sqrt{\lambda_i} \xi_i(\theta) \phi_i \quad (1.9)$$

with  $\xi_i$  independent standard normal variables and  $(\lambda_i, \phi_i)$  are the eigenvalues and eigenvectors of the covariance matrix  $\Sigma_{\chi\chi}$ .

#### 1.1.4 Karhunen-Loève expansion (KL expansion)

This method consist in expanding a random function, that is, a function defined on the  $\sigma$ -field of random events, in a Fourier-type series. It was derived independently by different researchers (Karhunen (1947) and Loeve (1948)) and is based on the spectral decomposition of the covariance function  $C_{HH}(\chi, \chi') = \sigma(\chi)\sigma(\chi')\rho(\chi, \chi')$ .

A process  $\chi(x)$  can be expanded into a series of the form

$$\hat{\chi}(x) = \sum_{n=1}^{\infty} \mathbf{c}_n f_n(x) \quad 0 < x < L \quad (1.10)$$

with  $f_n(x)$  a set of orthonormal functions in  $(0, L)$

$$\int_0^L f_n(x) f_m^*(x) dx = \delta[n - m] \quad (1.11)$$

The coefficients  $\mathbf{c}_n = \int_0^L \chi(x) f_n^*(x) dx$  are random variables. We consider the problem of determining a set of orthonormal functions  $f_n(x)$  such that the sum (1.10) equals  $\chi(x)$  and coefficients  $\mathbf{c}_n$  are orthogonal.

To solve this problem, the eigenfunctions  $f(x_1)$  and eigenvalues  $\lambda$  of the autocorrelation  $R(x_1, x_2)$  of the process  $\chi$ , are calculated solving the equation:

$$\int_0^L R(x_1, x_2) f(x_2) dx_2 = \lambda f(x_1) \quad 0 < x_1 < L \quad (1.12)$$

From theory of integral equations it is known that the eigenfunctions are orthonormal and satisfy the identity  $R(x, x) = \sum_{n=1}^{\infty} \lambda_n |f_n(x)|^2$ . Then it can be shown that

$$E\{|\chi(x) - \hat{\chi}(x)|^2\} = 0 \quad 0 < x < L \quad (1.13)$$

and

$$E\{\mathbf{c}_n \mathbf{c}_m^*\} = \lambda_n \delta[n - m] \quad (1.14)$$

We can note that if we have an orthonormal set of functions  $f_n(x)$  and

$$\chi(x) = \sum_{n=1}^{\infty} \mathbf{c}_n f_n(x) \quad (1.15)$$



$$\mathbb{E}\{\mathbf{c}_n \mathbf{c}_m^*\} = \begin{cases} \sigma_n^2 & n = m \\ 0 & n \neq m \end{cases} \quad (1.16)$$

Then the functions  $f_n(t)$  must satisfy equation (1.10) with  $\lambda = \sigma_n^2$ .

If the eigenfunctions are normalized so as to have  $\mathbb{E}\{\mathbf{c}_n \mathbf{c}_m^*\} = \delta_{nm}$  and we call the set of uncorrelated random variables with zero mean  $\{\xi_1, \dots, \xi_n\}$ , KL expansion becomes the well-known formula:

$$\chi(x) = \sum_{n=0}^{\infty} \xi_n \sqrt{\lambda_n} f_n(x) \quad (1.17)$$

The KL expansion has some interesting properties over other existing schemes (Sudret and Der-Kiureghian (2000)):

- It is error minimizing: the set of orthogonal functions given by this expansion is the one that gives the minimum mean-square error from a finite representation of the original process.
- KL expansion is unique.
- For a Gaussian process, KL expansion is almost surely convergent.

We consider the KL expansion of a random field with *exponential correlation function*

$$R(x_1, x_2) = e^{|x_1 - x_2|/b} \quad (1.18)$$

The eigenfunctions  $f$  and eigenvalues  $\lambda$  are the solution to equation

$$\int_{-a}^{+a} e^{|x_1 - x_2|/b} f(x_2) dx_2 = \lambda f(x_1) \quad (1.19)$$

Where  $b$  is the correlation length. The explicit expressions of the eigenfunctions and eigenvalues are given by:

$$\begin{cases} f_n(x) = \frac{\cos(\omega_n x)}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} & \lambda_n = \frac{2c}{\omega_n^2 + c^2} \quad \text{for } n \text{ odd} \\ f_n^*(x) = \frac{\sin(\omega_n^* x)}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} & \lambda_n^* = \frac{2c}{\omega_n^{*2} + c^2} \quad \text{for } n \text{ even} \end{cases} \quad (1.20)$$

where  $c = 1/b$  and  $\omega_n$  being the solutions of the first equation and  $\omega^*$  of the second:

$$\begin{cases} c - \omega \tan(\omega a) = 0 & \text{for } n \text{ odd} \\ \omega^* + c \tan(\omega^* a) = 0 & \text{for } n \text{ even} \end{cases} \quad (1.21)$$

## 1.2 Response statistics calculation for static problems

Now, the aim is to estimate the variance of the response, how disperse the response is around the mean. The input has  $N$  random variables related to geometry, material properties and loads. Each of them can be represented as the sum of its mean and a zero-mean random variables  $\alpha_i$ , the input variations around the mean are collected in a zero-mean random vector  $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_N\}$ . If the problem is not specifically dynamic, methods for solving the stochastic differential equation are based on second moment approaches and spectral representation.

### 1.2.1 Second moment approaches

Second moment methods evaluate the statistics of the output from the mean values of input variables and covariance matrix of  $\boldsymbol{\alpha}$ .

- The *perturbation method* uses Taylor series expansion of the quantities involved around their mean value: stiffness  $\mathbf{K}$ , response  $\mathbf{U}$  and load vector  $\mathbf{f}$ , and then identifies coefficients at both sides of the equation

$$\mathbf{K}\mathbf{U} = \mathbf{f} \quad (1.22)$$

The Taylor series expansion of each matrix is represented by:

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^N \mathbf{K}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathbf{K}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (1.23)$$

$$\mathbf{U} = \mathbf{U}_0 + \sum_{i=1}^N \mathbf{U}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathbf{U}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (1.24)$$

$$\mathbf{f} = \mathbf{f}_0 + \sum_{i=1}^N \mathbf{f}_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathbf{f}_{ij}^{II} \alpha_i \alpha_j + o(\|\boldsymbol{\alpha}\|^2) \quad (1.25)$$

with  $\mathbf{K}_i^I = \left. \frac{\partial \mathbf{K}}{\partial \alpha_i} \right|_{\alpha=0}$  and  $\mathbf{K}_{ij}^{II} = \left. \frac{\partial^2 \mathbf{K}}{\partial \alpha_i \partial \alpha_j} \right|_{\alpha=0}$

Identifying the similar order coefficients in equation (1.22) after substituting with equations (1.23) to (1.25), the following expressions are obtained

$$\mathbf{U}_0 = \mathbf{K}_0^{-1} \cdot \mathbf{f}_0 \quad (1.26)$$

$$\mathbf{U}_i^I = \mathbf{K}_0^{-1} \cdot (\mathbf{f}_i^I - \mathbf{K}_i^I \cdot \mathbf{U}_0) \quad (1.27)$$

$$\mathbf{U}_{ij}^{II} = \mathbf{K}_0^{-1} \cdot (\mathbf{f}_{ij}^{II} - \mathbf{K}_i^I \cdot \mathbf{U}_j^I - \mathbf{K}_j^I \cdot \mathbf{U}_i^I - \mathbf{K}_{ij}^{II} \cdot \mathbf{U}_0) \quad (1.28)$$

The statistics of  $\mathbf{U}$  are available from that of  $\boldsymbol{\alpha}$  in equation (1.24):

$$\mathbf{E}[\mathbf{U}] \approx \mathbf{U}_0 + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \mathbf{U}_{ij}^{II} \text{Cov}[\alpha_i, \alpha_j] \quad (1.29)$$

$$\text{Cov}[\mathbf{U}, \mathbf{U}] \approx \sum_{i=1}^N \sum_{j=1}^N \mathbf{U}_i^I \cdot (\mathbf{U}_j^I)^T \text{Cov}[\alpha_i, \alpha_j] \quad (1.30)$$

This method is used with a discretization of the random variables like spatial average method (Baecher and Ingra(1981), Vanmarcke and Grigoriu(1983)) or shape functions method (Liu *et al.* (1986a,b)).

- Another method based on perturbation method is the *weighted integrals method*. It couples perturbation method with the representation of the stochastic stiffness matrix as a random variable  $\mathbf{k}^e = \mathbf{k}_0^e + \sum_{l=1}^{NWI} \Delta \mathbf{k}_l^e \chi_l^e$  where  $\chi_l^e$  is the  $l$ -th of the *NWI* zero-mean weighted integrals of the random field and  $\Delta \mathbf{k}_l^e$  are deterministic matrices. As in the former case, statistics of the response can be calculated.
- The *quadrature method* is based on computing the moments of response quantities (Baldeweck (1999)). For a mechanical system, the uncertainties are described by a vector of uncorrelated normal random variables  $\mathbf{X} = \{X_1, \dots, X_N\}$  with prescribed joint distribution. If  $S$  is a response quantity, the  $i$ -th moment of  $S$  can be calculated with

$$\text{E}[S^i(X_1, \dots, X_N)] = \int_{\mathbb{R}^N} [s(x_1, \dots, x_N)]^i \varphi(x_1) \dots \varphi(x_N) dx_1 \dots dx_N \quad (1.31)$$

Where  $\varphi(\cdot)$  is the probability density function of a standard normal variate and  $s(x_1, \dots, x_N)$  is usually known. The quadrature of this integral is its approximation by a weighted summation of the function at some points of the interval, as in the Gauss-points method.

$$\text{E}[S^i(X_1, \dots, X_N)] \approx \sum_{k_1=1}^{NP} \dots \sum_{k_N=1}^{NP} w_{k_1} \dots w_{k_N} [s(x_{k_1}, \dots, x_{k_N})]^i \quad (1.32)$$

This method uses information about the distributions of the basic random variables for the calculations, so, it is not a "pure" second moment approach. This method does not discretize the random field and gives an upper bound on the response variance independent of the correlation structure of the field and second order statistics of the response.

- *Neumann expansion method* was introduced to the field of structural mechanics by Shinozuka and Nomoto (1980). The theory developed by Neumann states that the inverse of an operator (if exist) can be expanded in a convergent series in terms of the iterated kernels

$$[\mathbf{L} + \Pi] \mathbf{U} = \mathbf{F} \quad (1.33)$$

$$\mathbf{U} = [\mathbf{L} + \Pi]^{-1} \mathbf{F} \quad (1.34)$$

$$u(\alpha(\theta), \mathbf{x}) = \sum_{i=0}^{\infty} (-1)^i [\mathbf{L}^{-1}(\mathbf{x}) \Pi(\alpha(\mathbf{x}, \theta), \mathbf{x})]^i [f(\mathbf{x}, \theta)] \quad (1.35)$$

verifying that

$$\| \mathbf{L}^{-1} \mathbf{\Pi} \| < 1$$

If only one parameter ( $\alpha(\mathbf{x}, \theta)$ ) of our system is random, substituting it by its KL expansion  $\alpha(\mathbf{x}, \theta) = \sum_{n=1}^M \lambda_n \xi_n a_n(\mathbf{x})$  leads to:

$$\left[ \bar{\mathbf{K}} + \sum_{n=1}^M \xi_n \mathbf{K}^{(n)} \right] \mathbf{U} = \mathbf{f} \quad (1.36)$$

$$\left[ \mathbf{I} + \sum_{n=1}^M \xi_n \mathbf{Q}^{(n)} \right] \mathbf{U} = \mathbf{g} \quad (1.37)$$

where

$$\mathbf{Q}^{(n)} = \bar{\mathbf{K}}^{-1} \mathbf{K}^{(n)} \mathbf{g} = \bar{\mathbf{K}}^{-1} \mathbf{f} \quad (1.38)$$

$$[\mathbf{I} + \mathbf{\Psi} [\{\xi_n\}]] \mathbf{U} = \mathbf{g} \quad (1.39)$$

Where Neumann expansion is applied

$$\mathbf{U} = \sum_{i=0}^{\infty} (-1)^i \left[ \sum_{n=1}^M \xi_n \mathbf{Q}^{(n)} \right]^i \mathbf{g} \quad (1.40)$$

This expression is computationally more tractable than the original Neumann expansion.

### 1.2.2 Spectral Stochastic Finite Elements Method (SSFEM)

This method is an extension of the deterministic FEM for boundary value problems involving random material properties. In a deterministic case, a system motion is governed by a partial differential equation (PDE) with associated boundary conditions and initial conditions. The FEM discretizes those differential equations replacing the geometry by a set of nodes, and the displacement field is approximated by the nodal displacements

$$\mathbf{K} \mathbf{U} = \mathbf{F} \quad (1.41)$$

$$\mathbf{k}^e = \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega_e \quad (1.42)$$

where  $\mathbf{K}$ ,  $\mathbf{U}$ ,  $\mathbf{F}$ ,  $\mathbf{k}_e$ ,  $\mathbf{B}$  and  $\mathbf{D}$  are respectively the stiffness matrix, displacement vector, external force vector, element stiffness matrix, matrix relating strains to nodal displacements and elasticity matrix.

If system properties are modelled as random fields, the system is governed by a stochastic PDE and the response will be a displacement random field  $u(x, \theta)$  where  $\theta$  denotes a basic outcome in the space of all possible outcomes. It is discretized as a random vector of nodal displacements  $\mathbf{U}(\theta)$ . If a finite set of points  $\{\theta_1, \dots, \theta_Q\}$  is selected so as to sample correctly the space of all outcomes, it defines the random variable: this is the approach of Monte Carlo

simulation.

A different approach, taken by SSFEM, is to discretize the random field with series expansions. The input random field is discretized using a truncated KL expansion and each nodal displacement is represented by its coordinates in an appropriate basis of the space of random variables called *polynomial chaos* (PC).

Consider  $\{\xi_i(\theta)\}_{i=1}^{\infty}$ , a set of orthonormal Gaussian random variables. The set of polynomials  $\Gamma_p$  in  $\{\xi_i(\theta)\}_{i=1}^{\infty}$  of degree not exceeding  $p$ , orthogonal to the set of polynomials  $\Gamma_{p-1}$ , are the PC of order  $p$ . Those polynomials will be symmetrical with respect to their arguments. Any square-integrable random function can be approximated as closely as desired with the representation:

$$\begin{aligned} \mu(\theta) &= a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2} \sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) + \dots \\ &= \sum_{j=0}^{\infty} \hat{a}_j \Psi_j[\boldsymbol{\xi}(\theta)] \end{aligned} \quad (1.43)$$

The convergence properties of this expansion depends on the order of PC and number of terms in KL expansion. All PC are orthogonal, that means that

$$\int_{\Omega_e} \Gamma_i(\xi)\Gamma_j(\xi)dP = \langle \Gamma_i\Gamma_j \rangle = 0 \quad (1.44)$$

with  $\xi$  an element of the space of all possible outcomes representing a random variable with zero mean and  $P$  the probability of that outcome). The uncorrelated Gaussian random variables verify

$$\langle \xi_1, \dots, \xi_{2n+1} \rangle = 0 \quad \text{and} \quad \langle \xi_1, \dots, \xi_{2n} \rangle = \sum \prod \langle \xi_i \xi_j \rangle \quad (1.45)$$

Those conditions are the ones used for calculate them:

$$\Gamma_0 = 1 \quad \text{the first polynomial chaos is a constant} \quad (1.46)$$

$$\Gamma_1(\xi_i) = \xi_i \quad \text{the PC with order } > 0 \text{ have zero mean} \quad (1.47)$$

$$\Gamma_2(\xi_{i_1}, \xi_{i_2}) = \xi_{i_1}\xi_{i_2} - \delta_{i_1 i_2} \quad (1.48)$$

$$\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) = \xi_{i_1}\xi_{i_2}\xi_{i_3} - \xi_{i_1}\delta_{i_2 i_3} - \xi_{i_2}\delta_{i_1 i_3} - \xi_{i_3}\delta_{i_1 i_2} \quad (1.49)$$

and so on. After substituting PC expansion and KL expansion, the equilibrium equation becomes:

$$\left( \sum_{i=0}^{\infty} \mathbf{K}_i \xi_i(\theta) \right) \cdot \left( \sum_{j=0}^{\infty} \mathbf{U}_j \Psi_j(\theta) \right) - \mathbf{F} = 0 \quad (1.50)$$

After truncation of expansions, the error of the approximation is

$$\epsilon_{M,P} = \sum_{i=0}^M \sum_{j=0}^{P-1} \mathbf{K}_i \cdot \mathbf{U}_j \xi_i(\theta) \Psi_j(\theta) - \mathbf{F} \quad (1.51)$$

It can be minimized in a mean square sense:  $E[\epsilon_{M,P} \cdot \Psi_k] = 0 \quad k = 0, \dots, P-1$

if  $\mathbf{K}_{jk} = \sum_{i=0}^M \sum_{j=0}^{P-1} E[\xi_i \Psi_j \Psi_k] \mathbf{K}_i$  and  $\mathbf{F}_k = E[\Psi_k \mathbf{F}]$ , the error minimizing equation leads to the system

$$\begin{bmatrix} \mathbf{K}_{00} & \dots & \mathbf{K}_{0,P-1} \\ \mathbf{K}_{10} & \dots & \mathbf{K}_{1,P-1} \\ \vdots & & \vdots \\ \mathbf{K}_{P-1,0} & \dots & \mathbf{K}_{P-1,P-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{P-1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{P-1} \end{bmatrix} \quad (1.52)$$

Knowing that  $\mathbf{U}(\theta) = \sum_{j=0}^{P-1} \mathbf{U}_j \Psi_j(\theta)$ , the mean and covariance of nodal displacement vector can be calculated:

$$E[\mathbf{U}] = \mathbf{U}_0 \quad (1.53)$$

and

$$\text{Cov}[\mathbf{U}, \mathbf{U}] = \sum_{j=0}^{P-1} E[\Psi_j^2] \mathbf{U}_j \mathbf{U}_j^T \quad (1.54)$$

### 1.3 Response statistics calculation for dynamic problems

In a deterministic dynamic system, the solution  $\mathbf{U}$  is obtained by inverting the dynamic stiffness matrix

$$\mathbf{D}(s) = s^2 \mathbf{M} + s \mathbf{G}(s) + \mathbf{K} \quad (1.55)$$

That is, getting the transfer matrix  $\mathbf{H}(s) = \mathbf{D}^{-1}$ , and multiplying it by the forcing vector  $\mathbf{U} = \mathbf{H}\mathbf{f}$ . This procedure is simplified using the residues theorem, that states that any complex function can be expressed in terms of its poles  $s_j$  and residues. If the poles are the eigenvalues of  $\mathbf{H}(s)$ , the transfer matrix becomes

$$\mathbf{H}(s) = \sum_{j=1}^m \frac{\mathbf{R}_j}{s - s_j}. \quad (1.56)$$

Here

$$\mathbf{R}_j = \underset{s=s_j}{\text{res}} [\mathbf{H}(s)] \stackrel{\text{def}}{=} \lim_{s \rightarrow s_j} (s - s_j) [\mathbf{H}(s)] = \frac{\mathbf{z}_j \mathbf{z}_j^T}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j}. \quad (1.57)$$

is the residue of the transfer function matrix at the pole  $s_j$ , where  $\mathbf{z}_j$  is the corresponding eigenvector. The transfer function matrix may be obtained as

$$\mathbf{H}(i\omega) = \sum_{j=1}^N \left[ \frac{\gamma_j \mathbf{z}_j \mathbf{z}_j^T}{i\omega - s_j} + \frac{\gamma_j^* \mathbf{z}_j^* \mathbf{z}_j^{*T}}{i\omega - s_j^*} \right] + \sum_{j=2N+1}^m \frac{\gamma_j \mathbf{z}_j \mathbf{z}_j^T}{i\omega - s_j}, \quad (1.58)$$

where

$$\gamma_j = \frac{1}{\mathbf{z}_j^T \frac{\partial \mathbf{D}(s_j)}{\partial s_j} \mathbf{z}_j}. \quad (1.59)$$

Details about the derivation of this expressions can be found in Adhikari (2000).

Some special and useful cases can be defined:

1. *Undamped systems*: In this case  $\mathbf{G}(s) = 0$  results the order of the characteristic polynomial  $m = 2N$ ;  $s_j$  is purely imaginary so that  $s_j = i\omega_j$  where  $\omega_j \in \mathbb{R}$  are the undamped natural frequencies and  $\mathbf{z}_j = \mathbf{x}_j \in \mathbb{R}^N$ . In view of the mass normalization relationship  $\mathbf{K}\mathbf{x}_j = \omega_j^2 \mathbf{M}\mathbf{x}_j$ ,  $\gamma_j = \frac{1}{2i\omega_j}$  and equation (1.58) leads to

$$\mathbf{H}(i\omega) = \sum_{j=1}^N \frac{1}{2i\omega_j} \left[ \frac{1}{i\omega - i\omega_j} - \frac{1}{i\omega + i\omega_j} \right] \mathbf{x}_j \mathbf{x}_j^T = \sum_{j=1}^N \frac{\mathbf{x}_j \mathbf{x}_j^T}{\omega_j^2 - \omega^2}. \quad (1.60)$$

2. *Viscously-damped systems with non-proportional damping* (Lancaster (1966), Vigneron (1986), Géradin and Rixen (1997)): In this case  $m = 2N$  and  $\gamma_j = \frac{1}{\mathbf{z}_j^T [2s_j \mathbf{M} + \mathbf{C}] \mathbf{z}_j}$ . These reduce expression (1.58) to

$$\mathbf{H}(i\omega) = \sum_{j=1}^N \left[ \frac{\gamma_j \mathbf{z}_j \mathbf{z}_j^T}{i\omega - s_j} + \frac{\gamma_j^* \mathbf{z}_j^* \mathbf{z}_j^{*T}}{i\omega - s_j^*} \right]. \quad (1.61)$$

3. *Viscously-damped systems with proportional damping* derives from the former case, where the eigenvectors are real and  $s_j = -\zeta_j \omega_j + i\omega_j$ , so they coincide with their complex conjugate

$$\mathbf{H}(i\omega) = \sum_{j=1}^N \left[ \frac{\gamma_j \mathbf{z}_j \mathbf{z}_j^T}{-\omega^2 - 2i\omega \zeta_j \omega_j + \omega_j^2} \right] \quad (1.62)$$

The above expressions lead to the calculation of the response vector as a function of the system matrices eigenvalues and eigenvectors. Therefore, a study of the effect of randomness in eigenvalues and eigenvectors should lead to a study of the randomness in the displacement field.

### 1.3.1 Random eigenvalue problem

The determination of natural frequencies and mode shapes require the solution of an eigenvalue problem. This problem could be either a differential eigenvalue problem or a matrix eigenvalue problem depending on whether a continuous model or a discrete model is chosen to describe the given vibrating system. Several studies have been conducted on this topic since the mid-sixties: see Boyce (1968), Scheidt and Purkert (1983), Ibrahim (1987), Benaroya and Rehak (1988), and Benaroya (1992). Here a brief review of the literature on methods used for random eigenvalue problem of differential equations and matrices will be given.

### Continuous systems

Different models have been studied: the free vibration of elastic string and rod with random mass density and area of cross section were studied by Boyce (1962), the random string vibration problem by Goodwin and Boyce (1964), the case of random elastic beam and beam columns were considered by Boyce and Goodwin (1964), Haines (1967) and Bliven and Soong (1969), stochastic boundary value problems by Linde (1969) and Boyce (1966), beam column with random material and geometric properties with random axial load by Hoshiya and Shah (1971), etc. Widely used methods for the determination of eigenvalues and normal modes are variational, asymptotic and perturbation method, leading to a Sturm-Liouville problem. Also Monte Carlo simulation is used.

The free vibration characteristics of systems governed by second order stochastic wave equation have been investigated by Iyengar and Manohar (1989) and Manohar and Iyengar (1993): the probability distribution of eigenvalues of random string equation are related to the zeros of an associated initial value problem. Exact solutions to the problems are shown to be possible only under special circumstances (Iyengar and Manohar (1989) and Manohar and Keane (1993)) and, consequently, approximations become necessary. For specific types of mass and stiffness variations, Iyengar and Manohar (1989) and Manohar and Iyengar (1993, 1994) have developed solution strategies based on closure, discrete Markov chain approximation, stochastic averaging methods and Monte Carlo simulation to obtain acceptable approximations to the probability densities of the eigensolutions. A qualitative feature of the eigenvalue distributions emerged from these studies has been the tendency of the eigenvalues, when normalized with respect to the deterministic eigenvalues, to become stochastically stationary with respect to the mode count  $n$  for large values of  $n$ .

Studies on eigenvalues of non-self adjoint stochastic differential equation with stochastic boundary conditions have been conducted by Ramu and Ganesan (1993c) using perturbation method. They considered a random Leipholz column with distributed random tangential follower load and derived the correlation structure of the free vibration eigenvalues and critical loads in terms of second order moment properties of the system property random fields. Ganesan (1996) also considered stochastic non-self adjoint systems and obtained second order properties of the eigenvalues for different correlation structure of the system parameters.

### Discrete systems

For the above mentioned stochastic continuous systems, the applications are on relatively simple structures (*e.g.* single beam-column, axially vibrating rod, string, rectangular plate) and for simple boundary conditions. For the more general class of problems generally found in engineering, discretization methods, such as FEM, are adopted. Then the governing



eigenvalue problems have the form

$$\mathbf{K}(\Theta)\mathbf{X}(\Theta) = \lambda(\Theta)\mathbf{M}(\Theta)\mathbf{X}(\Theta) \quad (1.63)$$

Most of the available formulations are based on perturbation or Taylor series methods. Fox and Kapoor (1968) have presented exact expressions for the rate of change of eigenvalues and eigenvectors with respect to design parameters of the actual structure in terms of system property matrices and eigenvalue and eigenvector under consideration. If  $\delta_j$  ( $j = 1..r$ ), are design variables,  $\lambda_i$  and  $\mathbf{X}_i$  are  $i$ th eigenvalue and eigenvector respectively, then, the rate of change of eigenvalue can be expressed as

$$\frac{\partial \lambda_i}{\partial \delta_j} = \mathbf{X}_i^T \left[ \frac{\partial \mathbf{K}}{\partial \delta_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial \delta_j} \right] \mathbf{X}_i \quad (1.64)$$

They also presented two different formulation for rate of change of eigenvectors; these are as follows:

*Formulation 1:*

$$\frac{\partial \mathbf{X}_i}{\partial \delta_j} = - [\mathbf{F}_i \mathbf{F}_i + 2\mathbf{M} \mathbf{X}_i \mathbf{X}_i^T \mathbf{M}]^{-1} \left[ \mathbf{F}_i \frac{\partial \mathbf{F}_i}{\partial \delta_j} + \mathbf{M} \mathbf{X}_i \mathbf{X}_i \frac{\partial \mathbf{M}}{\partial \delta_j} \right] \mathbf{X}_i \quad (1.65)$$

$$\text{where } \mathbf{F}_i = \mathbf{K} - \lambda_i \mathbf{M} \quad (1.66)$$

$$(1.67)$$

*Formulation 2:*

$$\frac{\partial \mathbf{X}_i}{\partial \delta_j} = \sum_{k=1}^r a_{ijk} \mathbf{X}_k \quad (1.68)$$

$$\text{where } a_{ijk} = \frac{\mathbf{X}_l^T \left[ \frac{\partial \mathbf{K}}{\partial \delta_j} - \lambda_i \frac{\partial \mathbf{M}}{\partial \delta_j} \right] \mathbf{X}_i}{\lambda_l - \lambda_i} \quad \text{for } l \neq i \quad (1.69)$$

$$= -\frac{1}{2} \left[ \mathbf{X}_i^T \frac{\partial \mathbf{M}}{\partial \delta_j} \mathbf{X}_i \right] \quad \text{for } l = i \quad (1.70)$$

$$(1.71)$$

It may be noted that, formulation 1 requires the inversion of an  $n \times n$  matrix, whereas such an inversion does not appear in formulation 2. So formulation 2 is computationally less intensive and has been used by many authors later. Collins and Thomson (1969) have given expressions for the statistics of eigenvalues and eigenvectors in a matrix form in terms of mass and stiffness matrix of the structure. According to their formulation, covariance matrix for eigenvalue and eigenvector becomes

$$[Cov_{(\lambda,x)}] = \left[ \begin{array}{cc} \left[ \frac{\partial \lambda}{\partial k} \right] & \left[ \frac{\partial \lambda}{\partial m} \right] \\ \left[ \frac{\partial \lambda}{\partial x} \right] & \left[ \frac{\partial \lambda}{\partial m} \right] \end{array} \right] [Cov_{(k,m)}] \left[ \begin{array}{cc} \left[ \frac{\partial \lambda}{\partial k} \right] & \left[ \frac{\partial \lambda}{\partial x} \right] \\ \left[ \frac{\partial \lambda}{\partial x} \right] & \left[ \frac{\partial \lambda}{\partial m} \right] \end{array} \right]^T \quad (1.72)$$

Where the matrix  $[Cov_{(k,m)}]$  contains variance and covariance of stiffness and mass elements. They have also considered specific examples and compared the theoretical solutions with

results from Monte Carlo simulation. Schiff and Bogdanoff (1972a,b) derived an estimator for the standard deviation of a natural frequency in terms of second order statistical properties of system parameters. The derivation was based upon mean square approximate systems. Scheidt and Purkert (1983) have used perturbation expansion of eigenvalues and eigenvectors and have given the expression of homogeneous terms up to 4th order. These terms then can be used to determine expectation and correlation of the random eigenvalues and eigenvectors. Bucher and Brenner (1992) have employed a first order perturbation and, starting from the definition of Rayleigh's quotient for discrete systems, and, assuming  $\lambda_i = \lambda_{i0} + \lambda_{i1}$ ,  $x_i = x_{i0} + \epsilon x_{i1}$ ,  $K = K_0 + \epsilon K_1$  and  $M = M_0 + \epsilon M_1$ , they have shown that

$$E[\lambda_i] = \frac{x_{i0}^T K_0 x_{i0}}{x_{i0}^T M_0 x_{i0}} \quad (1.73)$$

and

$$\sigma_{\lambda_i}^2 = \epsilon^2 \lambda_{i0}^2 E\{(x_{i0}^T K_1 x_{i0})^2 + 2(x_{i0}^T K_1 x_{i0})(x_{i0}^T M_1 x_{i0}) + (x_{i0}^T M_1 x_{i0})^2\} \quad (1.74)$$

In these equations it is assumed that  $\epsilon \ll 1$  and subscripts containing a 0 denote deterministic quantities and subscripts with 1 denote random quantities. Zhu *et al.* (1992) have used method of local averages to discretize random fields in conjunction with perturbational approach to study the statistics of fundamental natural frequency of isotropic rectangular plates. Their formulation also allows for multiplicity of deterministic eigenvalues. The latter issue has also been addressed by Zhang and Chen (1991). Song *et al.* (1995) have outlined a first order perturbational approach to find the moments of the sensitivity of random eigenvalues with respect to expected value of specified design variables.

Studies on random matrix eigenvalue problems arising in the study of structural stability have been conducted by Ramu and Ganesan (1992, 1993a,b, 1994), Sankar *et al.* (1993) and Zhang and Ellingwood (1995) using perturbational approaches. The study of flutter of uncertain laminated plates with random modulus of elasticity, mass density, thickness, fibre orientation of individual lamina, geometric imperfection of the entire plate and in-plane loads was carried out by Liaw and Yang (1993). Based on the study of distribution of zeros of random polynomials, Grigoriu (1992) has examined the roots of characteristic polynomials of real symmetric random matrices. It is suggested that such studies serve to identify the most likely values of eigenvalues, determine the average number of eigenvalues within a specified range and, finally, in conjunction with an assumption that eigenvalues constitute Poisson points, to find the probability that there is no eigenvalue below or above a given value or within a given range. Furthermore, the assumption that the random coefficients of the polynomial are jointly Gaussian is shown to simplify the analysis significantly.

Solutions that do not consider the perturbation method around the deterministic natural frequencies values are studied in Adhikari (2007a), Adhikari and Friswell (2007) and Adhikari (2006). In these references is considered a joint probability distribution of the natural frequencies based on perturbation expansion of the eigenvalues about an optimal point and an

asymptotic approximation of multidimensional integrals. In Wagenknecht *et al.* (2005) possibilistic uncertainties are propagated through the coupled second-order differential equations that govern a dynamic system using pseudospectra of matrices.

### 1.3.2 High and mid frequency analysis: Non-parametric approach

This section focuses on the analysis of linear structural dynamical systems to which, at a given frequency, vibrational response consist of small contributions from a large number of modes, e.g. light flexible structures subjected to high frequency broad band excitation. The procedure used to analyze a structure, as described by Langley (1989), consist in:

1. Dividing the structure into a number of interacting subsystems whose mean energies and external power inputs are related via a set of linear equations. The coefficients of those linear equations are expressed in terms of quantities known as the loss factors and the coupling loss factors.
2. Study the systems individually, where the  $j$ th subsystem has a single response variable  $u_i(\mathbf{x}, t)$ , and  $\mathbf{x}$  represents the spatial co-ordinates of a point within the subsystem. The equation of motion which governs harmonic vibration of the  $j$ th subsystem is

$$(1 + i\varepsilon_j)L_j(u_j) - \rho_j\omega^2(1 - i\gamma_j)u_j = F_j(\mathbf{x}, \omega) + f_j^c(\mathbf{x}, \omega) \quad (1.75)$$

where  $\omega$  is the frequency of vibration,  $u_j(z\mathbf{x}, \omega)$  is the complex amplitude of the response,  $L_j$  is a differential operator depending on the structure (e.g. for a beam in bending  $L_j = EIu_j^{iv}$ ),  $\rho_j$  is the volume density (or equivalent) and  $\varepsilon_j$  and  $\gamma_j$  are dissipation factors.  $F_j(\mathbf{x}, \omega)$  represents an external load (a distributed force) while  $F_j^c(\mathbf{x}, \omega)$  is a load arising from coupling to another system.

3. Allow the systems to interact, which allows the exchange of vibration energies between the subsystems. The system response is characterized by calculating these energy exchanges. The mean external power input into subsystem  $j$  is

$$\hat{\mathbf{Q}}_j = 2\omega\varepsilon_j\hat{\mathbf{S}}_j + 2\omega\gamma_j\hat{\mathbf{T}}_j + \hat{\mathbf{R}}_j \quad (1.76)$$

where  $\hat{\mathbf{T}}_j$ ,  $\hat{\mathbf{S}}_j$  and  $\hat{\mathbf{R}}_j$  are the statistical average kinetic energy, the statistical average strain energy stored in the subsystem and the transmitted power. If  $\hat{\mathbf{E}} = \pi[\rho^{-1}q^{-1}]\hat{\mathbf{T}}$  then

$$\hat{\mathbf{Q}} = (2/\pi)\omega[\gamma\rho q]\hat{\mathbf{E}} + (1/\pi)\mathbf{N}\bar{\mathbf{M}}^{-1}[q]\hat{\mathbf{E}} \quad (1.77)$$

where the first term of the equation represents the power dissipated internally, while the second is the power transmitted.  $[\gamma]$  is a diagonal matrix of terms  $\gamma_j$ , dissipation factor of each subsystem  $j$ ,  $[\rho]$  has the same relation towards the volume density,  $[q]$ ,  $\mathbf{N}$  and  $\bar{\mathbf{M}}$  are matrices whose entries depend on the Green function  $G_{ij}(\mathbf{x}, \mathbf{y}, \omega)$  representing the response at  $\mathbf{x}$  on subsystem  $i$  to a load applied at  $\mathbf{y}$  on subsystem  $j$ .

In all those steps, the uncertainty is introduced by the Green function, depending on which Green function is taken, the method becomes Statistical Energy Analysis (SEA) or Gaussian Orthogonal Ensemble (GOE). A comparison between those two methods can be found in Langley and Brown (2004). It should be noted that the parameters variance has no influence in SEA and GOE.

### Statistical Energy Analysis (SEA)

Description of Statistical Energy Analysis (SEA) can be found in Lyon and Dejong (1995). Different assumptions are adopted in the application of this method:

- The vibrating structure is considered to be drawn from an ensemble of nominally identical systems.
- The subsystem eigensolutions are treated as having prescribed probability distributions: a set of Poisson points on the frequency axes. The natural frequencies are mutually independent and identically (uniformly) distributed in a given frequency bandwidth.
- The subsystem mode-shapes are deterministic.
- the power input and loss factors are uncorrelated.

The formulation presented by Lyon provides a fairly complete framework for obtaining measure of response variability. Those formulations were developed in early 1960's to enable prediction of vibration response of launch and payload structures to rocket noise at launch. Subsequently, the method has been used not only in aerospace industry, but also in land based vehicle design, ship dynamics, building acoustics, machinery vibration and in seismic analysis of secondary systems. Some examples of recent applications are related by Lai and Soong (1990) and Hynna *et al.* (1995).

Over the last decade there has been a resurgence of interest in this area of research and several important papers dealing with the theoretical development of this method have appeared, see, for example, the papers by Hodges and Woodhouse (1986), Langley (1989), Keane (1992) and Fahy (1994); also see the theme issue on SEA brought out by the Royal Society of London (Price and Keane (1994)) and the book edited by Keane and Price (1997).

### Gaussian Orthogonal Ensemble (GOE)

Work on random matrix theory has investigated in detail the statistical properties of the natural frequencies of a particular type of random matrix known as the Gaussian Orthogonal Ensemble (GOE) (see Mehta (1991)). The GOE concerns a symmetric matrix whose entries are Gaussian, zero mean, and statistically independent, with the off-diagonal entries having a common variance, and the diagonal entries having twice this variance. The spacings

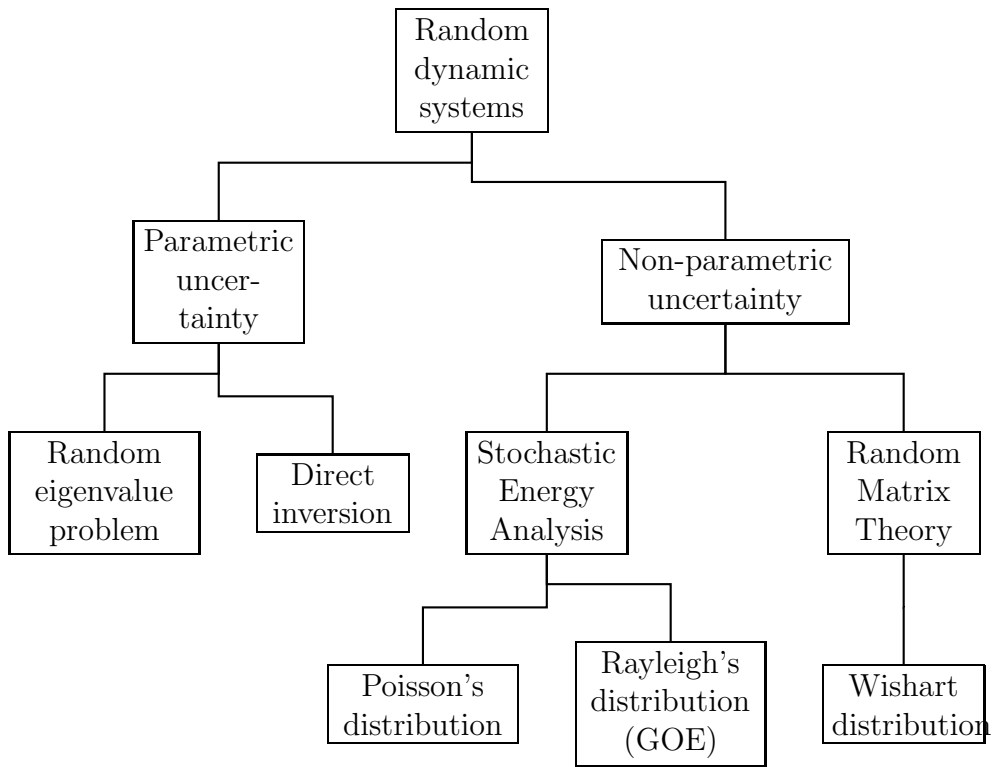
between the eigenvalues obtained with GOE have a Rayleigh distribution (referred to as the Wigner distribution in the random matrix literature). This contrasts with the exponential distribution that would be obtained were the natural frequencies to form a Poisson point process (used in SEA). It has arisen from numerical and experimental studies that most physical systems have natural frequencies that conform to the GOE statistics, even though the random matrices that govern their behaviour have no obvious direct connection with the GOE. However, when the system has many "symmetries" and the natural frequencies from all symmetric classes are superposed, the Poisson spacing statistics are obtained, see Weaver (1989), Ellegaard *et al.* (2001), Bertelsen *et al.* (2000) and Langley and Brown (2004).

### 1.3.3 High and mid frequency analysis: Random Matrix Theory

The Random Matrix Theory was firstly studied by physicists and mathematicians, in the 1930s, in the context of nuclear physics. Many useful results can be found in Mehta (1991) or Gupta and Nagar (2000). It has been said that SEA and GEO do not take account of parametric variance. Therefore, a parametric approach using random matrices is being investigated. Using the maximum entropy principle for positive symmetric real random matrices, Soize (2001) obtained an ensemble distribution of random matrices. The fitness of this approach to the problem at low-frequency range and a study of the statistics of the random eigenvalues of these random matrices were studied in Soize (2003). The probability density function of that ensemble distribution of random matrices coincide, under some conditions, with the one of Wishart distribution.

## 1.4 Summary

As a summary, available methods for solving a stochastic differential equation are represented in the following chart:



# Chapter 2

## Random Matrix Theory (RMT)

As it has already been discussed, a discretized mechanical system equation of equilibrium is given by  $\mathbf{DU} = \mathbf{f}$  where the dynamic stiffness matrix and force vector can be random and consequently the nodal displacement is also random. Until now, some methods have considered a parametric approach where randomness in stiffness matrix is a consequence of randomness in material properties (Young's modulus, mass density, Poisson's ratio, damping coefficient) and geometric parameters: all aleatoric uncertainties. Epistemic uncertainties were studied separately, they do not depend explicitly on the system parameters: they can be unquantified errors associated with the equation of motion (linear or non linear), damping model (viscous or non viscous), model of structural joints and numerical method. RMT approach could take account of these two different kinds of uncertainties.

### 2.1 Multivariate statistics

Multivariate statistics is a generalization of statistics to matrices. It is used widely in physics to model quantum systems where the probability density function of eigenvalues of random matrices is calculated. Several works in continuous matrix variate distribution theory have been published, like Gupta and Nagar (2000), Muirhead (1982) and Mehta (1991). The concept of random matrix derives from that of a random vector  $\mathbf{X} = (X_1, \dots, X_m)'$ : a vector whose components are jointly distributed, with defined mean or vector of expectations  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$

$$\boldsymbol{\mu} \equiv E(\mathbf{X}) = (E(X_1), \dots, E(X_m))' \quad (2.1)$$

$$\boldsymbol{\Sigma} \equiv \text{Cov}(\mathbf{X}) = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'] \quad (2.2)$$

A largely studied case is the case when this random vector is normally distributed, then the marginal distribution of each component of  $\mathbf{X}$  is univariate normal. If  $\mathbf{X}$  is now a  $p \times q$  matrix,  $\text{vec}(\mathbf{X})$  is the  $pq \times 1$  vector formed by stacking the columns of  $\mathbf{X}$  under each other. Then the distribution of a matrix is in fact the distribution of the vector  $\text{vec}(\mathbf{X})$ , and this matrix is a random matrix.

A matrix random phenomenon is an observable phenomenon represented in a matrix form where the outcome of different observations is not deterministically predictable, but they obey certain conditions of statistical regularity. The set of all outcomes of a random matrix characterize the sample space  $\Theta$ , where a probability function assigning a probability to each outcome can be defined, a probability density function and a moment generating function can as well be defined.

A random matrix  $\mathbf{X}(p \times n)$  is said to have matrix variate normal distribution with mean  $\mathbf{N}(p \times n)$  and covariance matrix  $\mathbf{\Sigma} \otimes \mathbf{\Psi}$   $\mathbf{X} \sim N_{p,n}(\mathbf{N}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$  where  $\mathbf{\Sigma}(p \times p) > 0$  and  $\mathbf{\Psi}(n \times n) > 0$ , if  $\text{vec}(\mathbf{X}') \sim N_{pn}(\text{vec}(\mathbf{N}'), \mathbf{\Sigma} \otimes \mathbf{\Psi})$ . Its probability density function is given by:

$$(2\pi)^{-\frac{1}{2}np} \det(\mathbf{\Sigma})^{-\frac{1}{2}n} \det(\mathbf{\Psi})^{-\frac{1}{2}p} \text{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X} - \mathbf{N})\mathbf{\Psi}^{-1}(\mathbf{X} - \mathbf{N}')\right\}, \quad \mathbf{X} \in \mathbb{R}^{p \times n}, \mathbf{N} \in \mathbb{R}^{p \times n} \quad (2.3)$$

A particular case of this random matrix is the symmetric matrix variate normal distribution. If  $\mathbf{X}(p \times p)$  is a symmetric matrix, the condition  $\mathbf{X} = \mathbf{X}'$  is verified, a  $\frac{1}{2}p(p-1) \times 1$  vector  $\text{vecp}(\mathbf{X}) = [x_{11} \ x_{12} \ x_{22} \ \dots \ x_{1p} \ \dots \ x_{pp}]$  with distribution  $N_{\frac{1}{2}p(p+1)}(\text{vecp}(\mathbf{N}), B_p'(\mathbf{\Sigma} \otimes \mathbf{\Psi})B_p)$  is defined and  $\mathbf{\Sigma}\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Sigma}$ . The probability density function of  $\mathbf{X}$  in terms of  $\text{vecp}(\mathbf{X})$

$$(2\pi)^{-\frac{1}{4}p(p-1)} \det(B_p'(\mathbf{\Sigma} \otimes \mathbf{\Psi})B_p)^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2}(\text{vecp}(\mathbf{X}) - \text{vecp}(\mathbf{N}))' \cdot B_p^+(\mathbf{\Sigma} \otimes \mathbf{\Psi})^{-1} B_p^+(\text{vecp}(\mathbf{X}) - \text{vecp}(\mathbf{N}))\right] \quad (2.4)$$

Where  $(B_p)_{ij,gh} = \frac{1}{2}(\delta_{ig}\delta_{jh} + \delta_{ih}\delta_{jg})$ , with  $i \leq p, j \leq p, g \leq h \leq p$  and  $B_p^+$  its Moore-Penrose inverse.

A Wishart distribution is a  $p$ -variate generalization of  $\chi^2$  distribution. A  $p \times p$  random symmetric positive definite matrix  $\mathbf{S}$  has Wishart distribution with parameters  $p, n$  and  $\mathbf{\Sigma}(p \times p) > 0$ , written as  $S \sim W_p(n, \mathbf{\Sigma})$ , if its p.d.f is given by:

$$\left\{2^{\frac{1}{2}np} \Gamma_p\left(\frac{1}{2}n\right) \det(\mathbf{\Sigma})^{\frac{1}{2}n}\right\}^{-1} \det(\mathbf{S})^{\frac{1}{2}(n-p-1)} \text{etr}\left(-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right), \mathbf{S} > 0, n \geq p \quad (2.5)$$

The mass and stiffness matrices  $\mathbf{M}$  and  $\mathbf{K}$  of a system are nonnegative definite symmetric matrices, the damping matrix  $\mathbf{C}$ , proportional to  $\mathbf{M}$  and  $\mathbf{K}$ , is also a nonnegative definite symmetric matrix. Therefore, stiffness and mass matrix could be modelled as Wishart matrices. The moments of the inverse of the dynamic stiffness matrix, the frequency response matrix  $\mathbf{H}(\omega) = \mathbf{D}^{-1}(\omega)$  (see (1.55)) should exist for all frequency  $\omega$ , that is  $E[\|\mathbf{H}(\omega)\|_{\mathbb{F}}^p] < \infty, \forall \omega$ . This condition will always be verified if the inverse moments of  $\mathbf{M}, \mathbf{K}$  and  $\mathbf{C}$  exist, conditions easier to analyze than the former one.

It was shown in Soize (2001) and Soize (2000) that given a symmetric nonnegative definite matrix  $\mathbf{G}$ , maximizing the entropy associated with the matrix variate probability density function  $p_{\mathbf{G}}(\mathbf{G})$

$$\mathcal{S}(p_{\mathbf{G}}) = - \int_{\mathbf{G} > 0} p_{\mathbf{G}}(\mathbf{G}) \ln \{p_{\mathbf{G}}(\mathbf{G})\} d\mathbf{G} \quad (2.6)$$



and using the equations that act like constraints to obtain  $p_{\mathbf{G}}$

$$\int_{\mathbf{G}_{>0}} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} = 1 \quad (\text{the normalization}) \quad (2.7)$$

$$\text{and } E[\mathbf{G}] = \int_{\mathbf{G}_{>0}} \mathbf{G} p_{\mathbf{G}}(\mathbf{G}) d\mathbf{G} = \overline{\mathbf{G}} \quad (\text{the mean matrix}) \quad (2.8)$$

The probability density function resulting from that analysis is the matrix variate gamma distribution. The main difference between the matrix variate gamma distribution and the Wishart distribution is that historically only integer values were considered for the shape parameter  $p$  in the Wishart matrices. Then, this  $p_{\mathbf{G}}$  is called in this thesis Wishart distribution probability density function with parameters  $p = (2\nu + n + 1)$  and  $\Sigma = \overline{\mathbf{G}}/(2\nu + n + 1)$ , that is  $\mathbf{G} \sim W_n(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1))$ .

## 2.2 Wishart matrix

It has already been said that a Wishart matrix is the generalization of  $\chi^2$  to matrices. If  $\mathbf{X}_{(n+1) \times p} \sim N(\mathbf{1}\boldsymbol{\mu}', \mathbf{I}_{n+1} \otimes \Sigma)$  is a random matrix defined as  $\mathbf{X} = [\mathbf{X}'_1 \dots \mathbf{X}'_{n+1}]'$  where  $\mathbf{X}_1, \dots, \mathbf{X}_{n+1}$  are independent  $N_p(\boldsymbol{\mu}, \Sigma)$  random vectors,  $\mathbf{1} = (1, \dots, 1)' \in \mathbb{R}^{n+1}$  and  $\Sigma$  is positive definite, a Wishart matrix can be considered as the distribution of the covariance of  $\mathbf{X}$  with  $n \geq p$ .

Then a Wishart matrix  $\mathbf{S} \sim W_p(n, \Sigma)$  has the same distribution as  $\mathbf{Z}'\mathbf{Z}$ , where the  $n \times p$  matrix  $\mathbf{Z}$  is  $N(\mathbf{0}, \mathbf{I}_n \otimes \Sigma)$  (the rows of  $\mathbf{Z}$  are independent  $N_p(\mathbf{0}', \Sigma)$  random vectors). The probability density function of  $\mathbf{S} \sim W_p(n, \Sigma)$  is given by:

$$\{2^{\frac{1}{2}np} \Gamma_p(\frac{1}{2}n) \det(\Sigma)^{\frac{1}{2}n}\}^{-1} \det(\mathbf{S})^{\frac{1}{2}(n-p-1)} \text{etr}(-\frac{1}{2}\Sigma^{-1}\mathbf{S}), \quad \mathbf{S} > 0, n \geq p$$

If we know the matrix to fit has Wishart distribution, the parameters of matrix  $\mathbf{Z}$  have to be obtained. The moments of  $\mathbf{G} \sim W_p(n, \Sigma)$  are given by Gupta and Nagar (2000) [Chapter 3]

$$E[\mathbf{G}] = p\Sigma = \overline{\mathbf{G}} \quad (2.9)$$

$$E[\mathbf{G}^2] = p\Sigma^2 + p\text{Trace}(\Sigma)\Sigma + p^2\Sigma^2 = \frac{1}{\theta + n + 1} \left[ (\theta + n + 2)\overline{\mathbf{G}}^2 + \overline{\mathbf{G}} \text{Trace}(\overline{\mathbf{G}}) \right] \quad (2.10)$$

$$\text{cov}(G_{ij}, G_{kl}) = p(\Sigma_{ik}\Sigma_{jl} + \Sigma_{il}\Sigma_{jk}) = \frac{1}{\theta + n + 1} (\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}) \quad (2.11)$$

$$E[\{\mathbf{G} - E[\mathbf{G}]\}^2] = E[\mathbf{G}^2] - \overline{\mathbf{G}}^2 = \frac{1}{\theta + n + 1} \left[ \overline{\mathbf{G}}^2 + \overline{\mathbf{G}} \text{Trace}(\overline{\mathbf{G}}) \right] \quad (2.12)$$

Soize (2000) introduced the normalized standard deviation of  $\mathbf{G}$  as

$$\sigma_G^2 = \frac{E[\|\mathbf{G} - E[\mathbf{G}]\|_{\text{F}}^2]}{\|E[\mathbf{G}]\|_{\text{F}}^2} \quad (2.13)$$

It can be shown, using (2.9) to (2.12) that

$$\sigma_G^2 = \frac{p\text{Trace}(\Sigma^2) + p\{\text{Trace}(\Sigma)\}^2}{p^2\text{Trace}(\Sigma^2)} = \frac{1}{p} \left\{ 1 + \frac{\{\text{Trace}(\Sigma)\}^2}{\text{Trace}(\Sigma^2)} \right\} \quad (2.14)$$

With  $p = 2\nu + n + 1$ , it is obvious that, for a system with dimensions  $n = \text{constant}$ , the bigger becomes the order of the inverse moment  $\nu$  the smaller is  $\sigma_G^2$ , a measure of the uncertainty in the system matrices. This parameter  $\nu$  is generally  $\nu = 2$ , but it could be chosen so as to decrease the uncertainty of the system, even if to do that would imply the order of the inverse moment constraint increases.

Interest on Wishart matrix  $\mathbf{G} \sim W_p(n, \mathbf{\Sigma})$  also arise from the fact that the inverted Wishart matrix distribution  $p_{\mathbf{V}} \sim IW_n(m, \mathbf{\Psi})$ , is available making use of the jacobian of the transformation  $\mathbf{V} = f(\mathbf{G}) = \mathbf{G}^{-1}$  and the exterior product of each of the differential vectors  $d\mathbf{G}$  and  $d\mathbf{V}$ . A  $n \times n$  symmetric positive definite random matrix  $\mathbf{V}$  is said to have inverted Wishart distribution with parameters  $m$  and  $\mathbf{\Psi}$

$$p_{\mathbf{V}}(\mathbf{V}) = \frac{2^{-\frac{1}{2}(m-n-1)n} \det\{\mathbf{\Psi}\}^{\frac{1}{2}(m-n-1)}}{\Gamma_n\left(\frac{1}{2}(m-n-1)\right) \det\{\mathbf{V}\}^{m/2}} \text{etr}\{-\mathbf{V}^{-1}\mathbf{\Psi}\}; \quad m > 2n, \mathbf{\Psi} > 0. \quad (2.15)$$

$$\mathbb{E}[\mathbf{G}^{-1}] = \frac{\mathbf{\Psi}}{m-2n-2} = \frac{\theta+n+1}{\theta} \overline{\mathbf{G}}^{-1} \quad (2.16)$$

$$\begin{aligned} \mathbb{E}[\mathbf{G}^{-2}] &= \frac{\text{Trace}(\mathbf{\Psi})\mathbf{\Psi} + (m-2n-1)\mathbf{\Psi}^2}{(m-2n-1)(m-2n-2)(m-2n-4)} \\ &= \frac{(\theta+n+1)^2 \text{Trace}(\overline{\mathbf{G}}^{-1})\overline{\mathbf{G}}^{-1} + \theta\overline{\mathbf{G}}^{-2}}{\theta(\theta+1)(\theta-2)} \end{aligned} \quad (2.17)$$

We can observe that, contrarily to what was supposed with Wishart matrix, the mean of inverse Wishart matrix is not equal to the inverse of the mean. In fact, knowing that  $\nu \geq 2$  and that matrices obtained with a FEM tend to represent big structures with thousands of degrees of freedom  $n$ , those two values  $\mathbb{E}[\mathbf{G}^{-1}]$  and  $\overline{\mathbf{G}}^{-1}$  can be very different.

## 2.3 Fitting of parameter for Wishart distribution

A measure of the normalized standard deviation was introduced by Soize (2000), it is the dispersion parameter  $\sigma_G^2$  from (2.13):

$$\sigma_G^2 = \frac{\mathbb{E}[\|\mathbf{G} - \mathbb{E}[\mathbf{G}]\|_{\text{F}}^2]}{\|\mathbb{E}[\mathbf{G}]\|_{\text{F}}^2} \quad (2.18)$$

For a Wishart distribution, as seen in Adhikari (2008)

$$\sigma_G^2 = \frac{1}{\theta+n+1} \left\{ 1 + \frac{\{\text{Trace}(\overline{\mathbf{G}})\}^2}{\text{Trace}(\overline{\mathbf{G}}^2)} \right\}$$

This parameter can also be obtained from the FEM if parameters which contain uncertainty are expanded as KL expansion. Then, matrix  $\mathbf{G} = \{\mathbf{K}, \mathbf{M}, \mathbf{C}\}$  can be expanded as

$$\mathbf{G} = \mathbf{G}_0 + \sum_i \xi_i \mathbf{G}_i \quad \text{with} \quad \mathbf{G}_0 = \mathbb{E}[\mathbf{G}] \quad (2.19)$$

Details of this expansion are shown in Chapter 3, Chapter 4 and Chapter 5 for particular cases of rod, beam and thin plate. Then, introducing this expansion in (2.18)

$$\sigma_G^2 = \frac{\text{E} [\|\sum_i \xi_i \mathbf{G}_i\|_F^2]}{\|\text{E} [\mathbf{G}]\|_F^2} \quad (2.20)$$

$$= \frac{\text{E} \left[ \text{Trace} \left( (\sum_i \xi_i \mathbf{G}_i) (\sum_j \xi_j \mathbf{G}_j) \right) \right]}{\|\mathbf{G}_0\|_F^2} \quad (2.21)$$

$$= \frac{\text{Trace} \left( \text{E} \left[ (\sum_i \sum_j \xi_i \xi_j \mathbf{G}_i \mathbf{G}_j) \right] \right)}{\|\mathbf{G}_0\|_F^2} \quad (2.22)$$

$$\sigma_G^2 = \frac{\text{Trace} \left( (\sum_i \mathbf{G}_i^2) \right)}{\|\mathbf{G}_0\|_F^2} \quad (2.23)$$

Several criteria can be developed to fit a symmetric random matrix with a Wishart matrix, four are explained here, each of them verifying a different condition related to the first moment and that the normalized standard deviation is same as the measured normalized standard deviation  $\sigma_G = \tilde{\sigma}_G$ :

1. *Criteria 1:* For each system matrices, we consider that the mean of the random matrix is same as the deterministic matrix  $\text{E} [\mathbf{G}] = \overline{\mathbf{G}}$ . This condition results

$$p = n + 1 + 2\nu \quad \text{and} \quad \Sigma = \overline{\mathbf{G}}/p \quad (2.24)$$

where

$$\nu = \frac{1}{2\tilde{\sigma}_G^2} \left\{ 1 + \frac{\{\text{Trace}(\overline{\mathbf{G}})\}^2}{\text{Trace}(\overline{\mathbf{G}}^2)} \right\} - \frac{(n+1)}{2} \quad (2.25)$$

This criteria was proposed by Soize (2000, 2001).

2. *Criteria 2:* Called the optimal Wishart distribution, it was proposed by Adhikari (2007b). The dispersion parameter is obtained after considering that the mean of the random matrix and the mean of the inverse of the random matrix are closest to the deterministic matrix and its inverse: that is  $\|\overline{\mathbf{G}} - \text{E} [\mathbf{G}]\|_F$  and  $\|\overline{\mathbf{G}}^{-1} - \text{E} [\mathbf{G}^{-1}]\|_F$  are minimum. This condition results:

$$p = n + 1 + 2\nu \quad \text{and} \quad \Sigma = \overline{\mathbf{G}}/\alpha \quad (2.26)$$

where  $\alpha = \sqrt{2\nu(n+1+2\nu)}$  and  $2\nu$  is as defined in equation (2.25). The rationale behind this approach is that a random system matrix and its inverse should be mathematically treated in a similar manner as both are symmetric and positive-definite matrices.

3. *Criteria 3:* Here we consider that for each system matrix, the mean of the inverse of the random matrix equals to the inverse of the deterministic matrix:  $\text{E} [\mathbf{G}^{-1}] = \overline{\mathbf{G}}^{-1}$

. Using equation (2.16) and after some simplifications one obtains

$$p = n + 1 + 2\nu \quad \text{and} \quad \Sigma = \overline{\mathbf{G}}/2\nu \quad (2.27)$$

where  $\nu$  is defined in equation (2.25).

4. *Criteria 4:* This criteria arises from the idea that the mean of the eigenvalues of the distribution is same as the ‘measured’ eigenvalues of the mean matrix and the normalized standard deviation is same as the measured normalized standard deviation. Mathematically we can express this as

$$\mathbb{E}[\mathbf{M}^{-1}] = \overline{\mathbf{M}}^{-1}, \quad \mathbb{E}[\mathbf{K}] = \overline{\mathbf{K}}, \quad \sigma_M = \tilde{\sigma}_M \quad \text{and} \quad \sigma_K = \tilde{\sigma}_K \quad (2.28)$$

This implies that the parameters of mass matrix is selected according to criteria 3 and the parameters for the stiffness matrix is selected according to criteria 1. The damping matrix is obtained using criteria 2. This case is therefore a combination of the previous three cases.

It is verified with numerical simulations that the third criteria gives the best results.

## 2.4 Relationship between dispersion parameter and the correlation length

As it has been seen, the parameter  $\sigma_G^2$  is of the form

$$\sigma_G^2 = \frac{\text{Trace}((\sum_i \mathbf{G}_i^2))}{\|\mathbf{G}_0\|_{\mathbb{F}}^2} \quad (2.29)$$

Where matrices  $\mathbf{G}_i$  are obtained by integration of the shape functions of the FE model chosen multiplied by the KL expansion of a stochastic parameter of the system. Eigenfunctions in KL expansion are always orthogonal, like the trigonometric functions obtained in subsection 1.1.4. A property of functions cosine and sine is that they are bounded: they vary between  $-1$  and  $1$ . Then:

$$|\cos(ax + ab)| \leq 1 \quad (2.30)$$

$$|\cos(ax + ab)f_j(x)| \leq |f_j(x)| \quad (2.31)$$

$$\left| \int \cos(ax + ab)f_j(x) dx \right| \leq \left| \int f_j(x) dx \right| \quad (2.32)$$

$$\left( \int_0^l \cos(ax + ab)f_j(x) dx \right)^2 \leq \left( \int_0^l f_j(x) dx \right)^2 \quad (2.33)$$

$$\sum_j \left( \int_0^l \cos(ax + ab)f_j(x) dx \right)^2 \leq \sum_j \left( \int_0^l f_j(x) dx \right)^2 \quad (2.34)$$

Where  $f_j$  is a shape function and the sum for all the shape functions can be considered as a sum of the square of the elements  $G_{ij}$  of the matrix  $\mathbf{G}$  for which the parameter  $\sigma_G$  is being calculated.

It is observed here that in fact, each term of the matrix can have contributions from the element matrices of different elements. It can be noticed that the right hand side of (2.34) remains the same for those elements: if  $f_n(x)$  and  $f_m(x)$  are contributions from two different element matrices,  $f_n(x) + f_m(x)$  can be considered as a new shape function. The left hand side of (2.34) has different trigonometric function for each element: if an element has a trigonometric function  $\cos(a(x + b_1))$  the next one has  $\cos(a(x + b_2))$ . The difference  $|b_1 - b_2|$  is equal to the distance between local origins of two different elements: it is the length of the element if all elements have same length. We can also notice that if the number of elements used is infinity, this difference  $|b_1 - b_2| = 0$ , and  $\cos(a(x + b_1)) = \cos(a(x + b_2))$ . Then, we can consider that

$$\left( \int_0^l \cos(ax + ab_n) f_n(x) dx + \int_0^l \cos(ax + ab_m) f_m(x) dx \right)^2 \approx \left( \int_0^l \cos(ax + ab_n) (f_n(x) + f_m(x)) dx \right)^2 \quad (2.35)$$

$$\left( \int_0^l \cos(ax + ab_n) f_n(x) dx + \int_0^l \cos(ax + ab_m) f_m(x) dx \right)^2 \leq \left( \int_0^l (f_n(x) + f_m(x)) dx \right)^2 \quad (2.36)$$

So, we can identify  $G_{0_{kj}}$  as the right hand side of the equation and  $G_{i_{kj}}$  as the left hand side of the equation. If we write the parameter  $\sigma_G$  as

$$\sigma_G^2 = \frac{\sum_i \frac{\lambda_i \sigma_\beta^2}{\alpha_i^2} \sum_{kj}^n G_{i_{kj}}^2}{\sum_{kj}^n G_{0_{kj}}^2} \leq \sum_i \frac{\lambda_i \sigma_\beta^2}{\alpha_i^2 \beta^2} \quad (2.37)$$

where  $\sum_{kj}^n$  represents the summation to all the terms of a matrix,  $\alpha_i$  and  $\lambda_i$  are given by the KL expansion,  $\beta$  is the parameter that has been expanded (e.g.  $EA$  for a bar stiffness matrix) and  $\sigma_\beta$  is the parameter variance, generally a percentage of  $\beta$ .

## 2.5 Simulation method for Wishart system matrices

The random matrix model for a system matrix is going to be a Wishart matrix with parameters  $p = n + 1 + 2\nu$  and  $\Sigma = \overline{\mathbf{G}}/2\nu$ , with  $\nu$  given in (2.25), that means, criteria 3 is used. The following different steps can now be followed to simulate with Wishart matrices for the probabilistic structural dynamics:

1. Find  $n$ , the dimension of the system matrices. It is equal to the dimension of the deterministic matrices  $\overline{\mathbf{G}} \equiv \{\overline{\mathbf{M}}, \overline{\mathbf{C}}, \overline{\mathbf{K}}\}$  obtained using the standard FEM. For complex engineering systems  $n$  can be in order of several thousands or even millions.

2. Obtain the normalized standard deviations or the 'dispersion parameters'  $\tilde{\sigma}_G \equiv \{\tilde{\sigma}_M, \tilde{\sigma}_C, \tilde{\sigma}_K\}$  corresponding to the system matrices, from experiment, experience or using the Stochastic FEM.
3. Calculate

$$\theta_G = 2\nu = \frac{1}{\tilde{\sigma}_G^2} \left\{ 1 + \frac{\{\text{Trace}(\overline{\mathbf{G}})\}^2}{\text{Trace}(\overline{\mathbf{G}}^2)} \right\} - (n+1), \quad \text{for } \mathbf{G} = \{\mathbf{M}, \mathbf{C}, \mathbf{K}\} \quad (2.38)$$

4. Approximate  $(n+1+\theta_G)$  to its nearest integer and call it  $p$ . That is  $p = [n+1+\theta_G]$ . This approximation would introduce negligible error. Create a  $n \times p$  matrix  $\tilde{\mathbf{X}}$  with Gaussian random numbers with zero mean and unit covariance i.e.,  $\tilde{\mathbf{X}} \sim N_{n,p}(\mathbf{O}, \mathbf{I}_n \otimes \mathbf{I}_p)$ .
5. Because  $\Sigma = \overline{\mathbf{G}}/\theta_G$  is a positive definite matrix, it can be factorized as  $\Sigma = \mathbf{\Gamma}\mathbf{\Gamma}^T$ . Using the matrix  $\mathbf{\Gamma} \in \mathbb{R}^{n \times n}$ , obtain the matrix  $\mathbf{X}$  using the linear transformation

$$\mathbf{X} = \mathbf{\Gamma}\tilde{\mathbf{X}} \quad (2.39)$$

Following theorem 2.3.10 in Gupta and Nagar (2000) it can be shown that  $\mathbf{X} \sim N_{n,p}(\mathbf{O}, \Sigma \otimes \mathbf{I}_p)$ .

6. Now obtain the samples of the Wishart random matrices  $W_n(n+1+\theta_G, \overline{\mathbf{G}}/\theta_G)$  as

$$\mathbf{G} = \mathbf{X}\mathbf{X}^T \quad (2.40)$$

Alternatively, MATLAB<sup>®</sup> command `wishrnd` can be used to generate the samples of Wishart matrices. MATLAB<sup>®</sup> can handle fractional values of  $(n+1+\theta_G)$  so that the approximation to its nearest integer in step 5 may be avoided.

7. Repeat the above process for the mass, stiffness and damping matrices and solve the equation of motion for each sample to obtain the response statistics of interest.

This procedure can be implemented easily. Once the samples of the system matrices are generated, the rest of the analysis is identical to any Monte Carlo simulation based approach. If one implements this approach in conjunction with a commercial FE software, unlike the SSFEM, the commercial software needs to be accessed only once to obtain the mean matrices. This simulation procedure is therefore 'non-intrusive'.

Although the random matrix based approach outlined above is very easy to implement, there are some limitations:

- The statistical crossed correlations between mass, stiffness and damping matrices cannot be considered in the present form. One must derive matrix variate joint probability density function for  $\mathbf{M}$ ,  $\mathbf{K}$  and  $\mathbf{C}$ .

- Unlike the deterministic matrices derived from FE discretisation, sample-wise sparsity may not be preserved for the random matrices.
- Only one variable, namely the normalized standard deviation  $\sigma_G$  defined in (2.18), is available to characterize uncertainty in a system matrix. This can be obtained, for example using standard system identification tools (e.g., modal identification) across the ensemble.
- Unlike the parametric uncertainty quantification methods, no computationally efficient analytical method (e.g., perturbation method, polynomial chaos expansion) is yet available for random matrix approach.

In the medium and high-frequency vibration of stochastic systems, where there is enough ‘mixing’ of the modes, the dynamic response is not very sensitive to the detailed nature of uncertainty in the system. In this situation, in spite of the limitations mentioned above, the random matrix approach provides an easy alternative to the conventional parametric approach. The numerical examples in the next chapters illustrates this fact.

## 2.6 Generalized beta distribution

There is a gross ‘oversimplification’ of the nature of the uncertainty by only considering one parameter  $\theta$  as the origin or randomness in a random matrix. Any  $n \times n$  symmetric matrix  $\mathbf{G}$ , supposing fully populated, can have  $N = n(n + 1)/2$  number of independent elements. Therefore, its covariance matrix is  $N^2 = n^2(n + 1)^2/4$  dimensional and symmetric. Which implies that it can have  $N(N + 1)/2 = n(n + 1)(n(n + 1) + 2)/8$  number of independent elements. Nonparametric approach, therefore, only offers a single parameter to quantify uncertainty which can potentially be expressed by  $n(n + 1)(n(n + 1) + 2)/8$  number of independent parameters. However, when very little information regarding the covariance tensor of  $\mathbf{G}$  is available, then a central Wishart distribution is the best that can be used to quantify uncertainty. If more reliable information regarding the covariance tensor of  $\mathbf{G}$  is available, then we need a matrix variate distribution which not only satisfies being symmetric and positive definite, but also must offer more parameters to fit the ‘known’ covariance tensor of  $\mathbf{G}$ . This is the central motivation of fitting a random matrix with a beta distribution.

As stated in Gupta and Nagar (2000), a  $p \times p$  random symmetric positive definite matrix  $X$  has generalized matrix variate beta type  $I$  distribution with parameters  $a, b, \Omega, \Psi$  denoted by  $X \sim GB_p^I(a, b, \Omega, \Psi)$  if its p.d.f. is given by

$$\frac{\det(X - \Psi)^{a - \frac{1}{2}(p+1)} \det(\Omega - X)^{b - \frac{1}{2}(p+1)}}{\beta_p(a, b) \det(\Omega - \Psi)^{(a+b) - \frac{1}{2}(p+1)}}, \quad \Psi < X < \Omega \quad (2.41)$$

Where  $\beta$  is the multivariate beta function given by

$$\beta_p(a, b) = \int_{0 < A < I_p} \det(A)^{a - \frac{1}{2}(p+1)} \det(I_p - A)^{b - \frac{1}{2}(p+1)} dA = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} = \beta(b, a) \quad (2.42)$$

where  $\text{Re}(a) > \frac{1}{2}(p-1)$  and  $\text{Re}(b) > \frac{1}{2}(p-1)$

$X \sim GB_p^I(a, b, \Omega, \Psi)$  can be derived from a matrix variate beta type  $I$  distribution  $U \sim B_p^I(a, b)$  with the formula  $X = (\Omega - \Psi)^{\frac{1}{2}}U(\Omega - \Psi)^{\frac{1}{2}} + \Psi$ . Furthermore, the type  $I$  beta distribution can be obtained from independent  $Y \sim N_{p, n_1}(0, \Sigma \otimes I_{n_1})$  and  $S_2 \sim W_p(n_2, \Sigma)$  random matrices, with  $S = S_2 + YY'$ ,  $Z = S^{-\frac{1}{2}}Y$  and finally  $ZZ' \sim B_p^I(\frac{1}{2}n_1, \frac{1}{2}n_2)$  if  $n_1 \geq p$  and  $X = Z'Z \sim B_{n_1}^I(\frac{1}{2}p, \frac{1}{2}(n_1 + n_2 - p))$  if  $n_1 < p$ .

The system matrices are symmetric positive definite, they could be fitted with a generalized beta distribution. For this, assume  $\Psi = 0$ , parameters  $n_1$  and  $n_2$  and matrices  $\Omega$  (one for the stiffness matrix and another for mass matrix) have to be found.

Some properties about the generalized beta distribution  $X \sim GB_p^I(\frac{1}{2}n_1, \frac{1}{2}n_2; \Omega, 0)$  are:

$$\text{E}[x_{ij}] = \frac{n_1}{n} w_{ij} \quad (2.43)$$

$$\text{E}[x_{ij}x_{kl}] = \frac{n_1}{n(n-1)(n+2)} [(n_1(n+1) - 2)w_{ij}w_{kl} + n_2(w_{jl}w_{ki} + w_{il}w_{kj})] \quad (2.44)$$

$$\text{cov}(x_{ij}, x_{kl}) = \frac{n_1 n_2}{n(n-1)(n-2)} \left[ -\frac{2}{n} w_{ij}w_{kl} + w_{ij}w_{kl} + w_{il}w_{kj} \right] \quad (2.45)$$

For approximating the matrices with a beta distribution, we can state that the mean of the system matrices is equal to the mean of the generalized beta distribution matrix that approximates it, mathematically, and using (2.43):

$$\mathbf{K}_0 = \text{E}[X_K] \quad \mathbf{M}_0 = \text{E}[X_M] \quad (2.46)$$

$$\mathbf{K}_0 = \frac{n_1}{n} \Omega_K \quad \mathbf{M}_0 = \frac{n_1}{n} \Omega_M \quad (2.47)$$

This will allow us to find matrices  $\Omega_K$  and  $\Omega_M$  after finding  $n_1$  and  $n_2$ . Those two constants can be found identifying the mean of the terms of the covariance matrix of the beta distribution derived from (2.45):

$$\text{cov}(\text{vec}(\mathbf{U})) = \frac{n_1 n_2}{n(n-1)(n-2)} \left( -\frac{2}{n} \text{vec} \Omega \text{vec} \Omega' + \Omega \otimes \Omega + \mathbf{K}_{mn} \Omega \otimes \Omega \right) \quad (2.48)$$

An approach to the problem using beta distribution has not been found in the available literature, the Wishart method offers then an already tested frame for our study. On the other hand, the calculation of matrix  $\text{cov}(\text{vec}(\mathbf{U}))$  is computationally expensive, and simulations of random matrices following the beta model is not available on MATLAB<sup>®</sup>. The simulation of beta matrices based on MATLAB<sup>®</sup> available random matrices is also computationally expensive. The simulation using beta distribution is therefore not going to be used in this study.



# Chapter 3

## Axial vibration of rods with stochastic properties

The rod model is a 1D model that uses linear shape functions and is therefore one of the simplest FE models. It is also a widely used model, as many structural elements can be modeled as rods. Here the theory for deriving this model is reminded at first: the derivation of the equation of motion and the discretization process that leads to the well known deterministic FE model of the rod. Then, expressions for other mass and stiffness matrices taking account of the eigenfunctions appearing in the KL expansion of random variables  $EA$  and  $\rho A$  are derived.

### 3.1 Equation of motion

In this section, the governing equation of the system is derived, more details can be found in Petyt (1998) and G eradin and Rixen (1997). Hamilton's principle states

$$\int_{t_1}^{t_2} (\delta(T - U) + \delta W_{nc}) dt = 0 \quad (3.1)$$

Where  $T$  is the kinetic energy,  $U$  is the elastic strain energy,  $W_{nc}$  is the work of non conservative forces and  $\delta u = 0$  at  $t = t_1$  and  $t = t_2$ . In this study, linear constitutive model is assumed (the stress-strain relation is linear ( $\sigma_x = E\varepsilon_x$ , where the axial strain component  $\varepsilon_x = \frac{\partial u}{\partial x}$  and  $E$  is Young's modulus)), and the material is assumed as isotropic. It is assumed that each plane cross-section remains plane during the motion and the only stress component that is not negligible is  $\sigma_x$ , and it remains uniform over each cross section. The axial force in one of the faces of the increment  $dx$  of axial element is  $\sigma_x A$ , and the extension of the element is  $\varepsilon_x dx$ , the work done on the element as strain energy is  $dU = \frac{1}{2} \sigma_x \varepsilon_x A dx$  and for the complete rod, the total strain and kinetic energy are

$$U = \frac{1}{2} \int_{-a}^{+a} \sigma_x \varepsilon_x A dx = \frac{1}{2} \int_{-a}^{+a} EA \varepsilon_x^2 dx \quad (3.2)$$

and

$$T = \frac{1}{2} \int_{-a}^{+a} \rho A \dot{u}^2 dx \quad (3.3)$$

If there is an applied load of magnitude  $p_x$  per unit length, the work done in a virtual displacement  $\delta u$  is  $\delta u \cdot p_x dx$  and the virtual work for the whole rod is:

$$\delta W = \int_{-a}^{+a} p_x \delta u dx \quad (3.4)$$

After substituting those expressions in Hamilton's principle and assuming that operators  $\delta$  and  $\partial()/\partial t$  as well as  $\delta$  and  $\partial()/\partial x$  are commutative, and that integrations with respect to  $t$  and  $x$  are interchangeable, we can obtain the equation of motion:

$$EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} + p_x = 0 \text{ throughout } -a \leq x \leq +a \quad (3.5)$$

And boundary conditions

$$EA \frac{\partial u}{\partial x} = 0 \text{ or } \delta u = 0 \text{ at } x = -a \text{ and } x = +a \quad (3.6)$$

Depending on the boundary conditions, analytical solutions to this differential equation are available.

## 3.2 Deterministic Finite Element model

The basic rod element of length  $l$  has two nodes, one at each end, and one degree of freedom per node: the axial displacement  $u$  at each node, so, there are two degrees of freedom per element:  $u_1$  and  $u_2$ . The only forces acting on the element are point axial forces  $f_1$  and  $f_2$  acting at the ends (Dawe (1984)).

A displacement field, expressed in terms of the element nodal displacements, is assumed: for each point of the element the displacement is given by  $u = A_0 + A_1 x$ , or, in a matrix form

$$u \equiv \mathbf{u} = \boldsymbol{\alpha} \mathbf{A} = [1 \quad x] \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \quad (3.7)$$

If the origin of the  $x$  coordinate is at node 1, the boundary conditions  $\begin{cases} u = u_1 & \text{at } x = 0 \\ u = u_2 & \text{at } x = l \end{cases}$  for equation (3.7) lead to the system

$$\mathbf{d} = \mathbf{C} \mathbf{A} \quad \text{where} \quad \mathbf{A} = \mathbf{C}^{-1} \mathbf{d} \quad (3.8)$$

$$\mathbf{u} = \boldsymbol{\alpha} \mathbf{C}^{-1} \mathbf{d} = \mathbf{N} \mathbf{d} \quad (3.9)$$

or, expressed with matrices,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \quad \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (3.10)$$

and

$$\mathbf{N} = \left[ 1 - \frac{x}{l} \quad \frac{x}{l} \right] \quad (3.11)$$

Where  $\mathbf{N}$  is the *matrix of shape functions* or *shape function matrix*, and each shape function equals one for a given degree of freedom and zero for all the others.

In a discrete system, Hamiltons principle can be expressed as Lagranges equation (here stated without damping force):

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{u}} \right) - \frac{\partial U}{\partial u} = f \quad (3.12)$$

For this problem,  $f = p_x$  the axial force and the only strain and stress existing are  $\varepsilon_x$  and  $\sigma_x = E\varepsilon_x$  (acting in the axial direction).

$$\varepsilon_x = \frac{du}{dx} = \left[ \frac{d\mathbf{N}}{dx} \right] \mathbf{d} = \left[ -1/l \quad 1/l \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{B}\mathbf{d} \quad (3.13)$$

It is known that the (stretching) strain energy of the bar element of length  $l$  with origin placed in node one is:

$$U_e = \frac{1}{2} \int_0^l \sigma_x^t \varepsilon_x A dx = \frac{1}{2} \mathbf{d}^t \left( \int_0^l \mathbf{B}^t (AE) \mathbf{B} dx \right) \mathbf{d} \quad (3.14)$$

The element strain energy can be expressed as  $U_e = \frac{1}{2} \mathbf{d}^t \mathbf{K}_e \mathbf{d}$ , where  $\mathbf{K}_e$  is the element stiffness matrix. The stress is expressed as  $\sigma_x^t$  instead of  $\sigma_x$  due to the fact that while  $\sigma_x^t = \sigma_x$  and the expression of  $U_e$  remains the same scalar quantity, the element stiffness matrix can now be obtained and is independent of the displacement. The element stiffness matrix is given by:

$$\mathbf{K}_e = \int_0^l \mathbf{B}^t (AE) \mathbf{B} dx = AE \int_0^l \begin{bmatrix} -1/l \\ 1/l \end{bmatrix} \begin{bmatrix} -1/l & 1/l \end{bmatrix} dx = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.15)$$

$$\text{So,} \quad U = \frac{AE}{2l} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (3.16)$$

The work of the forces applied to the element is

$$W_e = -f_1 u_1 - f_2 u_2 \quad (3.17)$$

The kinetic energy of the rod element can be calculated in the same way

$$T_e = \frac{1}{2} \int_0^l \rho A \dot{u}^2 dx = \frac{1}{2} \dot{\mathbf{u}}^t \int_0^l \rho A \mathbf{N}^t \mathbf{N} dx \dot{\mathbf{u}} = \frac{1}{2} \dot{\mathbf{u}}^t \mathbf{M}_e \dot{\mathbf{u}} \quad (3.18)$$

$$\mathbf{M}_e = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (3.19)$$

After assembling the whole structure stiffness matrix, mass matrix and the forcing vector, those expressions can be introduced in Lagrange's equation (3.12) to obtain the system equation of motion:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \quad (3.20)$$

The assembly process consist in adding the element matrix corresponding to element number  $e$  to the submatrix composed of rows and columns  $e$  and  $e + 1$  of an initially zero matrix of dimensions  $N + 1$  where  $N$  is the number of elements of the rod structure. This process is done for mass and stiffness matrices, there is a similar process for the forcing vector where the terms of the element forcing vector are added to the corresponding terms of a global forcing vector.

### 3.3 Derivation of stiffness matrices

For the FE model of a straight axially loaded bar, the stiffness matrix is given by

$$\mathbf{K}_e = \int_0^l \mathbf{B}^t (AE) \mathbf{B} dx \quad (3.21)$$

where  $E$  is Youngs modulus,  $A$  denotes the cross section area, and the product  $AE$ , axial rigidity, is a random field that can be represented as a KL expansion

$$\mathbf{K}_e = \int_0^l \mathbf{B}^t (A_0 E_0 + \sum_{i=1}^{n/2} (\lambda_i f_i(x) \xi_i + \lambda_i^* f_i^*(x) \xi_i^*)) \mathbf{B} dx \quad (3.22)$$

with  $\xi_i$  and  $\xi_i^*$  set of uncorrelated random variables forming a Gaussian vector,  $\lambda_i$  and  $\lambda_i^*$  eigenvalues and  $f_i$  and  $f_i^*$  eigenfunctions of a given covariance function. The two different sets of eigenfunctions and corresponding eigenvalues have to be integrated for the covariance functions studied in subsection 1.1.4 As it has already been said:

$$c - \omega \tan(\omega a) = 0 \quad \text{gives a set of solutions } \omega_i \text{ for } i \text{ odd} \quad (3.23)$$

$$\omega + \omega \tan(\omega a) = 0 \quad \text{gives a set of solutions } \omega_i^* \text{ for } i \text{ even} \quad (3.24)$$

The eigenfunctions and eigenvectors can be calculated:

$$\lambda_i = \frac{2c}{\omega_i^2 + c^2} \quad \lambda_i^* = \frac{2c}{\omega_i^{*2} + c^2} \quad (3.25)$$

$$f_i(x) = \frac{\cos(\omega_i x)}{\sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} \quad f_i^*(x) = \frac{\sin(\omega_i^* x)}{\sqrt{a - \frac{\sin(2\omega_i^* a)}{2\omega_i^*}}} \quad (3.26)$$

The expression for the stiffness matrix considering KL expansion is:

$$\mathbf{K}_e = \mathbf{K}_{e_0} + \sum_{i=1}^{n/2} (\xi_i \lambda_i \mathbf{K}_{e_i} + \xi_i^* \lambda_i^* \mathbf{K}_{e_i}^*) \quad (3.27)$$

with

$$\mathbf{K}_0 = \frac{A_0 E_0}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.28)$$

$$\mathbf{K}_{e_i} = \frac{1}{l^2 \sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^l \cos(\omega_n(x + x_1^e)) dx \quad (3.29)$$

$$\mathbf{K}_{e_i}^* = \frac{1}{l^2 \sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^l \sin(\omega_n^*(x + x_1^e)) dx \quad (3.30)$$

noting that:

$$\int \cos(ax + b) dx = \int \cos(X) \frac{dX}{a} = \frac{\sin(X)}{a} = \frac{\sin(ax + b)}{a} \quad (3.31)$$

$$\int \sin(ax + b) dx = \int \sin(X) \frac{dX}{a} = \frac{-\cos(X)}{a} = -\frac{\cos(ax + b)}{a} \quad (3.32)$$

$$(3.33)$$

and finally,

$$\mathbf{K}_{e_i} = \frac{\sin(\omega_i(l + x_1^e)) - \sin(\omega_i x_1^e)}{l^2 \omega_i \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.34)$$

$$\mathbf{K}_{e_i}^* = \frac{-\cos(\omega_i^*(l + x_1^e)) + \cos(\omega_i^* x_1^e)}{l^2 \omega_i^* \sqrt{a - \frac{\sin(2\omega_i^* a)}{2\omega_i^*}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.35)$$

We can notice here that, contrarily to the deterministic stiffness matrix, the terms in the matrix will change from one element to the next one not only because of change in a given parameter ( $EA$ ), but also because of the effect of the eigenfunctions. This effect is neglected when undertaking a Monte Carlo simulation, and can only be reflected adopting a fine mesh.

### 3.4 Derivation of mass matrices

For the mass matrix, a similar process gives the symmetric matrices  $\mathbf{M}_{e_i}$  and  $\mathbf{M}_{e_i}^*$ , the components of whom are given by:

$$\mathbf{M}_{e_{i11}} = \frac{(2\omega_i l - 2 \sin(\omega_i l)) \cos(\omega_i x_1^e) + (-l^2 \omega_i^2 + 2 - 2 \cos(\omega_i l)) \sin(\omega_i x_1^e)}{l^2 \omega_n^3 \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}}$$

$$\mathbf{M}_{e_{i12}} = \frac{(-\cos(\omega_i l) l \omega_i + 2 \sin(\omega_i l) - \omega_i l) \cos(\omega_i x_1^e)}{l^2 \omega_n^3 \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} + \frac{(2 \cos(\omega_i l) + \omega_i l \sin(\omega_i l) - 2) \sin(\omega_i x_1^e)}{l^2 \omega_n^3 \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}}$$

$$\mathbf{M}_{e_{i22}} = \frac{(2 \cos(\omega_i l) l \omega_i + (l^2 \omega_i^2 - 2) \sin(\omega_i l)) \cos(\omega_i x_1^e)}{l^2 \omega_n^3 \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} + \frac{((l^2 \omega_i^2 - 2) \cos(\omega_i l) - 2 \omega_i l \sin(\omega_i l) + 2) \sin(\omega_i x_1^e)}{l^2 \omega_n^3 \sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}}$$

$$\begin{aligned}
\mathbf{M}_{e_{i11}^*} &= \frac{(l^2\omega_i^{*2} - 2 + 2\cos(\omega_i^*l))\cos(\omega_i^*el) + (2\omega_i^*l - 2\sin(\omega_i^*l))\sin(\omega_i^*el)}{l^2\omega_i^{*3}\sqrt{a - \frac{\sin(2\omega_i^*x)}{2\omega_i^*}}} \\
\mathbf{M}_{e_{i12}^*} &= \frac{(-2\cos(\omega_i^*l) + 2 - \sin(\omega_i^*l)l\omega_i^*)\cos(\omega_i^*el)}{l^2\omega_i^{*3}\sqrt{a - \frac{\sin(2\omega_i^*x)}{2\omega_i^*}}} + \\
&\quad \frac{(2\sin(\omega_i^*l) - \omega_i^*l - \omega_i^*l\cos(\omega_i^*l))\sin(\omega_i^*el)}{l^2\omega_i^{*3}\sqrt{a - \frac{\sin(2\omega_i^*x)}{2\omega_i^*}}} \\
\mathbf{M}_{e_{i22}^*} &= \frac{((-l^2\omega_i^{*2} + 2)\cos(\omega_i^*l) + 2\sin(\omega_i^*l)l\omega_i^* - 2)\cos(\omega_i^*el) +}{l^2\omega_i^{*3}\sqrt{a - \frac{\sin(2\omega_i^*x)}{2\omega_i^*}}} \\
&\quad \frac{(2\omega_i^*l\cos(\omega_i^*l) + (-2 + l^2\omega_i^{*2})\sin(\omega_i^*l))\sin(\omega_i^*el)}{l^2\omega_i^{*3}\sqrt{a - \frac{\sin(2\omega_i^*x)}{2\omega_i^*}}}
\end{aligned}$$

Those matrices are used in the following section to calculate the theoretical value of the dispersion parameter.

### 3.5 Numerical illustrations

The example model verifies that its boundary conditions are the ones of a clamped-free bar: zero displacement on the left end. Heuristically, it is considered that this boundary condition is the one that will more likely cause a bigger variance in the response vector than the one obtained with other typical boundary conditions like clamped-clamped bar. The material properties are taken as  $\rho = 2700 \text{ kg/m}^3$ ,  $E = 69 \text{ GPa}$ , values corresponding to aluminium. Geometrical variables are: length of the bar  $L = 30 \text{ m}$  and cross section  $A = 1 \text{ cm}^2$ . The variance of the parameters considered as random is 10% the value of the parameter:  $\sigma_\beta = 0.1\beta$ . The parameters considered are  $\beta = EA$  for stiffness matrix and  $\beta = \rho A$  for mass matrix. The system response is calculated using equation (1.62): equation for a viscously damped proportional model where damping factor is  $\zeta_j = 0.02$ . For choosing the number of elements of the structure, we consider that the smaller wavelength produced by the vibration of the dynamic structure should be at least four times the element length. If the maximum frequency for which the structure is analyzed covers the first fifty modes (number of modes  $k = 50$ ), the structure model will have 200 elements. This maximum frequency is then calculated with an analytical expression available for rod:

$$f_{max} = (2k - 1)\frac{1}{4}\sqrt{\frac{EA}{\rho AL^2}} = 4.17 \text{ KHz}$$

The maximum frequency used for testing the structure in this numerical calculation is  $4 \text{ KHz}$ . Measure of the response vector is given for the forcing point, situated at the 3rd node, and at a sensor situated at the 35-th node.

Four different correlation lengths used in the KL expansions are selected to research the effect of this value on the response statistics. Those correlation lengths are  $0.1L$ ,  $0.2L$ ,  $0.5L$

and  $L$ , where  $L$  is the bar total length. Also, the response statistics for uncorrelated random variables is calculated. The number of terms in each KL expansion is chosen such that the ratio between the square roots of the first term and last term eigenvalues is less than 0.05:  $\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_N}} \leq 0.05$

Correlation length	$0.1L$	$0.2L$	$0.5L$	$L$
Terms of KL expansion	67	36	18	12

Table 3.1: Number of terms used in KL expansion for each correlation length

In Figure 3.1 and Figure 3.2 we show the mean of the dynamic response at two points on the bar. From these figures it is observed that the mean of the absolute value of the response does not coincide with the response of the deterministic system with mean values of stochastic parameters. Also, the larger is the correlation length, the more different are the mean responses from the deterministic response. This observation can be done for both sensors where the system response is collected.

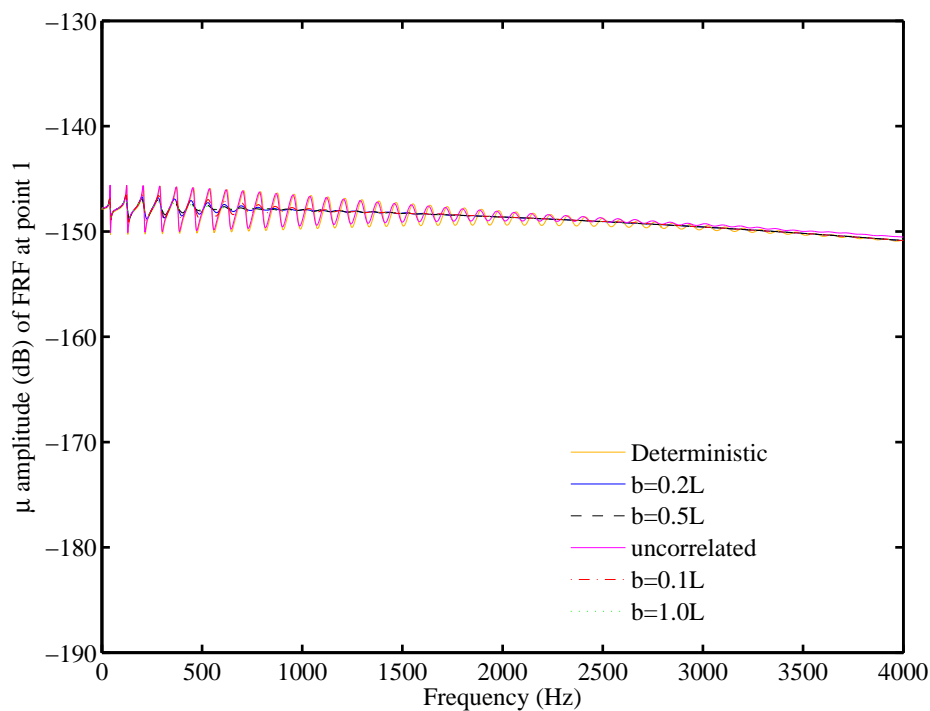


Figure 3.1: Absolute value of the Frequency Response Function (FRF) for the mean of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1, free end).

In Figure 3.3 and Figure 3.4 we show the standard deviation of the dynamic response at two points on the bar. A significant impact of the correlation of the random field can be observed on the dynamic response.

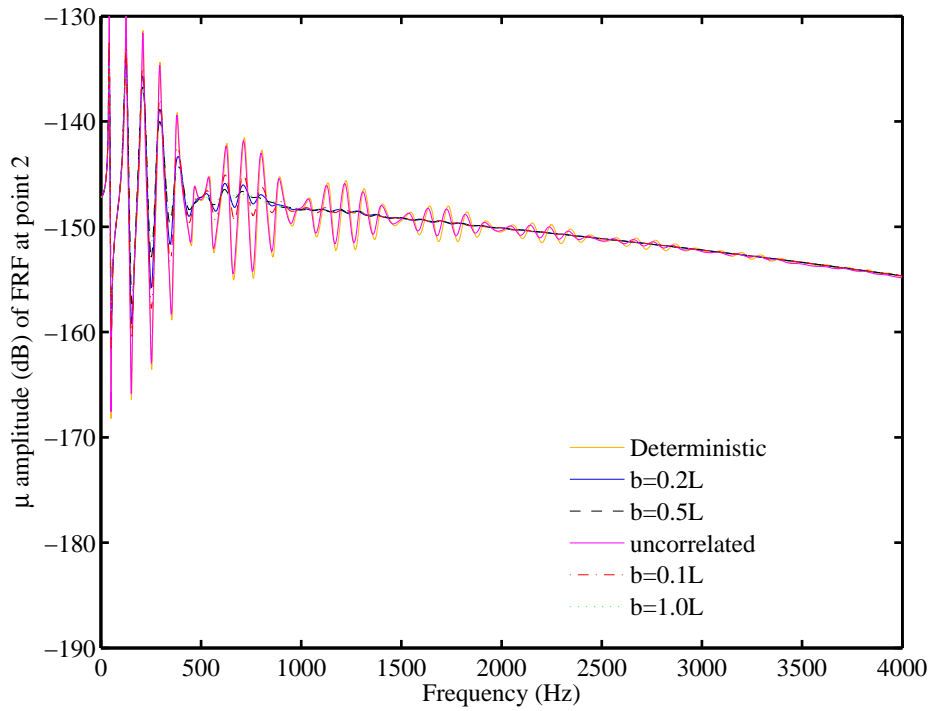


Figure 3.2: Absolute value of the Frequency Response Function (FRF) for the mean of the Monte Carlo simulations done directly with FEM for the sensor (Point 2, mid point).

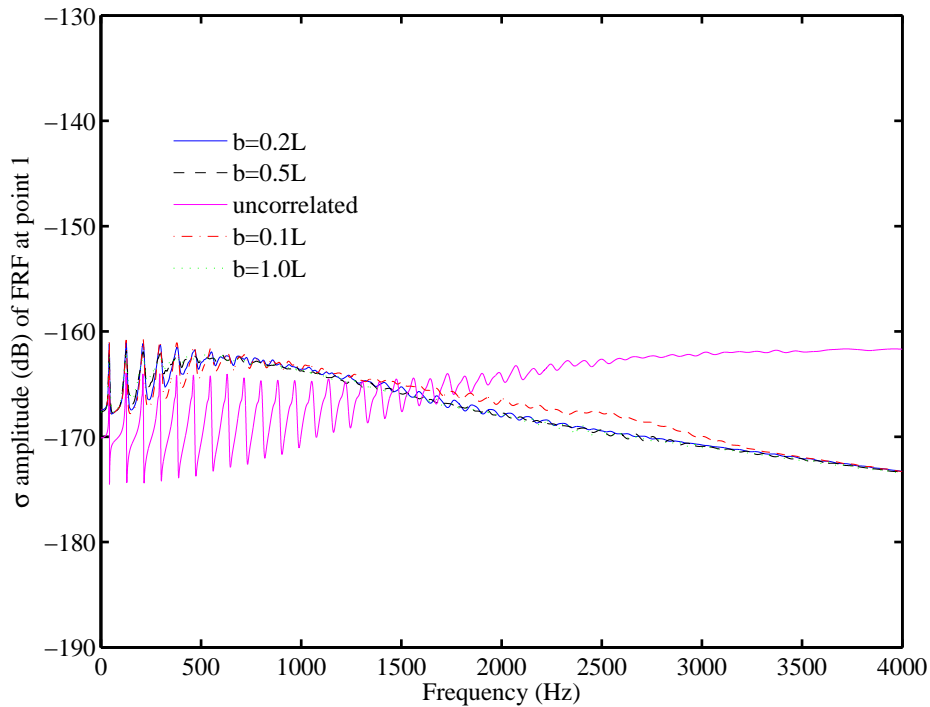


Figure 3.3: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1, free end) .



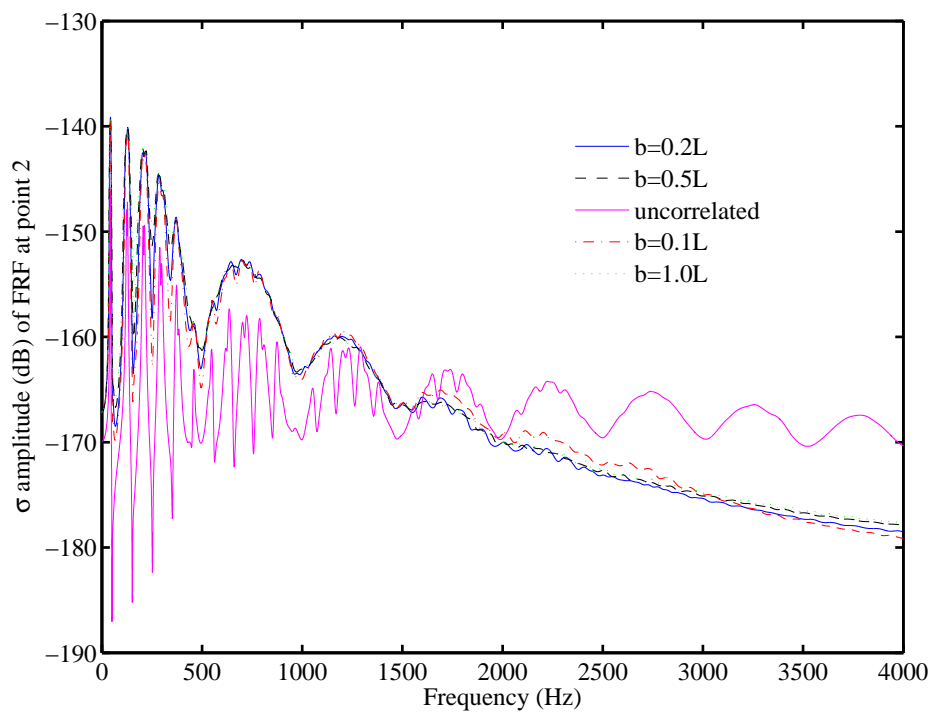


Figure 3.4: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the sensor (Point 2).

# Chapter 4

## Bending vibration of beams with stochastic properties

### 4.1 Equation of motion

As in the previous chapter, expressions for total strain energy  $U$ , kinetic energy  $T$ , and virtual work  $\delta W$  are derived, adopting the same hypothesis about linearity. The elementary theory of bending assumes that the stress components  $\sigma_y$ ,  $\sigma_z$ ,  $\tau_{yz}$  and  $\tau_{xz}$  are zero, that the plane sections normal to the undeformed centroidal axis remain plane after bending and that they remain normal to the deformed axis. A distributed load  $p_y$  per unit length is applied. Then, the axial displacement  $u$  at a distance  $y$  from the centroidal axis is  $u(x, y) = -y \frac{\partial v}{\partial x}$  where  $v$  is the displacement of the centroidal axis in the  $y$ -direction at position  $x$ . Expressions for strain and kinetic energy and virtual work are:

$$U = \int_V \sigma_x \varepsilon_x dv = \int_V E \varepsilon_x^2 dv = \int_{-a}^{+a} EI^2 \left( \frac{\partial^2 v}{\partial x^2} \right)^2 dx \quad (4.1)$$

$$\text{with } I_z = \int_A y^2 dA \quad (4.2)$$

$$T = \frac{1}{2} \int_{-a}^{+a} \rho A \dot{v}^2 dx \quad (4.3)$$

$$\delta W = \int_{-a}^{+a} p_y \delta v dx \quad (4.4)$$

After substituting those expressions in Hamilton's principle equation (3.1), with  $\delta v = 0$  when  $t = t_1$  and  $t = t_2$ , the equation of motion is obtained:

$$-EI \frac{\partial^2 v}{\partial x^4} - \rho A \frac{\partial^2 v}{\partial t^2} + p_y = 0 \quad \text{Throughout} \quad -a \leq x \leq +a \quad (4.5)$$

In addition, either

$$\begin{cases} EI_z \frac{\partial^2 v}{\partial x^2} = 0 & \text{or} & \frac{\partial v}{\partial x} = 0 \\ EI_z \frac{\partial^3 v}{\partial x^3} = 0 & \text{or} & v = 0 \end{cases} \quad (4.6)$$

at  $x = -a$  and  $x = +a$  act as boundary conditions.  $\frac{\partial v}{\partial x} = 0$  and  $v = 0$  represent the vanishing of slope and displacement respectively, while  $EI_z \frac{\partial^2 v}{\partial x^2} = 0$  and  $EI_z \frac{\partial^3 v}{\partial x^3} = 0$  represent the vanishing of stress over a cross-section and shear force.

## 4.2 Deterministic Finite Element model

The basic beam element of length  $l$  has two nodes, one at each end, and two degrees of freedom per node: the lateral displacement  $v$  and the rotation  $\theta_z \equiv \frac{dv}{dx}$  at each node, the degrees of freedom for one element are  $v_1, v_2, \theta_{z_1}$  and  $\theta_{z_2}$ . At each node of the element are acting a moment and a shear force as external forces:  $M_{z_1}, M_{z_2}, v_1$  and  $v_2$ .

The displacement field, expressed in terms of the element coordinates, is assumed to have a cubic form: for each point of the element the displacement is given by

$$\mathbf{u} \equiv v = A_0 + A_1x + A_2x^2 + A_3x^3 \quad (4.7)$$

$$= [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \quad (4.8)$$

$$\mathbf{u} = \boldsymbol{\alpha} \mathbf{A} \quad (4.9)$$

If the origin of the  $x$  coordinate is at node 1, the boundary conditions

$$\begin{cases} v = v_1 \text{ and } \theta_z = \theta_{z_1} & \text{at } x = 0 \\ v = v_2 \text{ and } \theta_z = \theta_{z_2} & \text{at } x = l \end{cases} \quad (4.10)$$

for equation (4.9) lead to the system

$$\mathbf{d} = \mathbf{C} \mathbf{A} \quad \text{where} \quad \mathbf{A} = \mathbf{C}^{-1} \mathbf{d} \quad (4.11)$$

and

$$\mathbf{u} = \boldsymbol{\alpha} \mathbf{C}^{-1} \mathbf{d} = \mathbf{N} \mathbf{d} \quad (4.12)$$

or, expressed in matrix form

$$\mathbf{d} = \begin{bmatrix} v_1 \\ \theta_{z_1} \\ v_2 \\ \theta_{z_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} \quad (4.13)$$

$$\mathbf{N} = \left[ 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \quad x - \frac{2x^2}{l} + \frac{x^3}{l^2} \quad \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad -\frac{x^2}{l} + \frac{x^3}{l^2} \right] \quad (4.14)$$

For the beam problem,  $f = p_y$  the transversal force. The only strain component is the axial strain

$$\varepsilon_x = -y \frac{d^2 v}{dx^2} = y \chi_z \quad (4.15)$$

where  $y$  is the distance from the neutral axis and  $\chi_z$  is the curvature and can be regarded as a ‘pseudo-strain’ while  $M_z$  can be regarded as a ‘pseudo stress’ and  $M_z = EI_z\chi_z$ . the strain energy of the element can be stated also as

$$U_e = \frac{1}{2} \int_0^l M_z^t \chi_z dz = \frac{1}{2} \int_0^l \chi_z^t EI_z \chi_z dz \quad (4.16)$$

The pseudo stress can be calculated

$$\chi_z = -\frac{d^2v}{dx^2} = -\left[\frac{d^2}{dx^2}\mathbf{N}\right] \mathbf{d} = \mathbf{B}\mathbf{d} \quad (4.17)$$

where

$$\mathbf{B} = \left[ \frac{6}{l^2} - \frac{12x}{l^3} \quad \frac{4}{l} - \frac{6x}{l^2} \quad \frac{-6}{l^2} + \frac{12x}{l^3} \quad \frac{2}{l} - \frac{6x}{l^2} \right] \quad (4.18)$$

Then, the strain energy

$$U_e = \frac{1}{2} \mathbf{d}^t \left( \int_0^l \mathbf{B}^t (EI_z) \mathbf{B} dx \right) \mathbf{d} = \frac{1}{2} \mathbf{d}^t \mathbf{K}_e \mathbf{d} \quad (4.19)$$

with

$$\mathbf{K}_e = \int_0^l \mathbf{B}^t (EI_z) \mathbf{B} dx \quad (4.20)$$

$$\mathbf{K}_e = EI_z \begin{bmatrix} 12/l^3 & 6/l^2 & -12/l^3 & 6/l^2 \\ & 4/l & -6/l^2 & 2/l \\ & \text{symmetric} & 12/l^3 & -6/l^2 \\ & & & 4/l \end{bmatrix} \quad (4.21)$$

The work of the forces applied to the element is

$$W_e = \int_0^l p_y \delta v dx = \delta \mathbf{u}^t \int_0^l p_y \mathbf{N}^t dx \quad (4.22)$$

The kinetic energy of the rod element can be calculated in the same way

$$T_e = \frac{1}{2} \int_0^l \rho A \dot{u}^2 dx = \frac{1}{2} \dot{\mathbf{u}}^t \int_0^l \rho A \mathbf{N}^t \mathbf{N} dx \dot{\mathbf{u}} = \frac{1}{2} \dot{\mathbf{u}}^t \mathbf{M}_e \dot{\mathbf{u}} \quad (4.23)$$

with

$$\mathbf{M}_e = \int_0^l \mathbf{N}^t (\rho A) \mathbf{N} dx \quad (4.24)$$

$$\mathbf{M}_e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & \text{symmetric} & 156 & -22l \\ & & & 4l^2 \end{bmatrix} \quad (4.25)$$

After assembling the whole structure stiffness matrix, mass matrix and the forcing vector, those expressions can be introduced in Lagrange’s equation (3.12) to obtain the system equation of motion:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \quad (4.26)$$

The assembly process of  $\mathbf{K}$ ,  $\mathbf{M}$  and  $\mathbf{f}$  is similar to the one described for rod.

### 4.3 Derivation of stiffness matrices

As expressed in the derivation of the FEM, the strain energy can be expressed as:

$$U_p = \frac{1}{2} \int_0^l M_z^t \chi_z dz = \frac{1}{2} \int_0^l \chi_z^t EI_z \chi_z dx = \frac{1}{2} \mathbf{d}^T \left( \int_{x=0}^l \mathbf{B}^t (EI_z) \mathbf{B} dx \right) \mathbf{d} \quad (4.27)$$

Where  $EI_z$  is the parameter that is going to be replaced by the usual KL expansion:

$$\mathbf{k} = \int_{x=0}^l \mathbf{B}^t (EI_z) \mathbf{B} dx \quad (4.28)$$

$$\mathbf{k} = \int_{x=0}^l \mathbf{B}^t \left( I_{z_0} E_0 + \sum_{i=1}^{n/2} (\lambda_i f_i(x) \xi_i + \lambda_i^* f_i^*(x) \xi_i^*) \right) \mathbf{B} dx \quad (4.29)$$

$$\mathbf{k} = \mathbf{k}_0 + \sum_{i=0}^{n/2} (\xi_i \lambda_i \mathbf{k}_i + \xi_i^* \lambda_i^* \mathbf{k}_i^*) \quad (4.30)$$

Matrices  $\mathbf{k}_0$ ,  $\mathbf{k}_i$  and  $\mathbf{k}_i^*$  are symmetric, and each of them has 16 terms: for each of them only 10 terms will have to be calculated.

$$\mathbf{k}_0 = E_0 I_{z_0} \begin{bmatrix} 12/l^3 & & & & \\ 6/l^2 & 4/l & \text{symmetric} & & \\ -12/l^3 & -6/l^2 & 12/l^3 & & \\ 6/l^2 & 2/l & -6/l^2 & 4/l & \end{bmatrix} \quad (4.31)$$

For integrating  $\mathbf{k}_i$  and  $\mathbf{k}_i^*$ , we observe that the coordinates for the eigenfunctions are linked to the whole structure, and not to each element, whereas functions in  $\mathbf{B}$  are linked to each element. Integration will be done in element coordinates, and variables of eigenfunctions are going to be changed: if  $X$  is the coordinate used by the eigenfunction, and  $x$  the one used in a particular element,  $X = x + x_1^e$ , with  $x_1^e$  being the coordinate of the initial node of the element, related to the whole structure.

$$\mathbf{k}_{i11} = \frac{36}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ (8 - \omega_n^2 l^2) (\sin(\omega_n x_1^e) - \sin(\omega_n (l + x_1^e))) + 4\omega_n l (\cos(\omega_n (l + x_1^e)) + \cos(\omega_n x_1^e)) \right]$$

with  $\mathbf{k}_{i11} = \mathbf{k}_{i33} = -\mathbf{k}_{i13}$

$$\mathbf{k}_{i12} = \frac{12}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ \sin(\omega_n (l + x_1^e)) (l^3 \omega_n^2 - 12l) + 2l \sin(\omega_n x_1^e) (6 - \omega_n^2 l^2) + l^2 \omega_n (5 \cos(\omega_n (l + x_1^e)) + 7 \cos(\omega_n x_1^e)) \right]$$

with  $\mathbf{k}_{i12} = -\mathbf{k}_{i32}$

$$\mathbf{k}_{i14} = \frac{12}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ 2l \sin(\omega_n (l + x_1^e)) (\omega_n^2 l^2 - 6) + l \sin(\omega_n x_1^e) (12 - \omega_n^2 l^3) + l^2 \omega_n (7 \cos(\omega_n (l + x_1^e)) + 5 \cos(\omega_n x_1^e)) \right]$$

with  $\mathbf{k}_{i14} = -\mathbf{k}_{i34}$

$$\mathbf{k}_{i22} = \frac{4}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ \sin(\omega_n(l + x_1^e)) (l^4 \omega_n^2 - 18l^2) + 2l^2 \sin(\omega_n x_1^e) (9 - 2l^2 \omega_n^2) + l^3 \omega_n (\cos(\omega_n(l + x_1^e)) + 12 \cos(\omega_n x_1^e)) \right]$$

$$\mathbf{k}_{i24} = \frac{4}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ 2l^2 (2\omega_n^2 - 9) (\sin(\omega_n(l + x_1^e)) - \sin(\omega_n x_1^e)) + 9l^3 \omega_n (\cos(\omega_n(l + x_1^e)) + \cos(\omega_n x_1^e)) \right]$$

$$\mathbf{k}_{i44} = \frac{4}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n} l^6 \omega_n^3}} \left[ 2l^2 \sin(\omega_n(l + x_1^e)) (2l^2 \omega_n^2 - 9) + l^2 \sin(\omega_n x_1^e) (18l^2 - 4l^2 \omega_n) + 6l^3 \omega_n (2 \cos(\omega_n(l + x_1^e)) + \cos(\omega_n x_1^e)) \right]$$

$$\mathbf{k}_{i11}^* = \frac{36}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^6 \omega_n^{*3}}} \left[ 4l\omega_n^* (\sin(\omega_n^*(l + x_1^e)) + \sin(\omega_n^* x_1^e)) + (l^2 \omega_n^{*2} - 8) (\cos(\omega_n^* x_1^e) - \cos(\omega_n^*(l + x_1^e))) \right]$$

$$\mathbf{k}_{i12}^* = \frac{12}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^5 \omega_n^{*3}}} \left[ \omega_n^* l (5 \sin(\omega_n^*(l + x_1^e)) + 7 \sin(\omega_n^* x_1^e)) + \cos(\omega_n^* x_1^e) (12 - l^2 \omega_n^{*2}) + 2 \cos(\omega_n^*(l + x_1^e)) (\omega_n^{*2} l^2 - 6) \right]$$

$$\mathbf{k}_{i14}^* = \frac{12}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^6 \omega_n^{*3}}} \left[ l\omega_n^* (7 \sin(\omega_n^*(l + x_1^e)) + 5 \sin(\omega_n^* x_1^e)) + \cos(\omega_n^* x_1^e) (l^2 \omega_n^{*2} - 12) + 2 \cos(\omega_n^*(l + x_1^e)) (6 - \omega_n^{*2} l^2) \right]$$

$$\mathbf{k}_{i22}^* = \frac{4}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^4 \omega_n^{*3}}} \left[ 6l\omega_n^* (\sin(\omega_n^*(l + x_1^e)) + 2 \sin(\omega_n^* x_1^e)) + 2 (2l^2 \omega_n^{*2} - 9) \cos(\omega_n^* x_1^e) + (18 - \omega_n^{*2} l^2) \cos(\omega_n^*(l + x_1^e)) \right]$$

$$\mathbf{k}_{i24}^* = \frac{2}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^4 \omega_n^{*3}}} \left[ 9l\omega_n^* (\sin(\omega_n^*(l + x_1^e)) + \sin(\omega_n^* x_1^e)) + 2 (l^2 \omega_n^{*2} - 9) (\cos(\omega_n^* x_1^e) - \cos(\omega_n^*(l + x_1^e))) \right]$$

$$\mathbf{k}_{i44}^* = \frac{4}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*} l^4 \omega_n^{*3}}} \left[ 6l\omega_n^* (2 \sin(\omega_n^*(l + x_1^e)) + \sin(\omega_n^* x_1^e)) + \cos(\omega_n^* x_1^e) (l^2 \omega_n^{*2} - 72) + 2 \cos(\omega_n^*(l + x_1^e)) (9 - 2\omega_n^{*2} l^2) \right]$$

and identities  $\mathbf{k}_{i11} = \mathbf{k}_{i33} = -\mathbf{k}_{i13}$ ,  $\mathbf{k}_{i12}^* = -\mathbf{k}_{i32}^*$ , and  $\mathbf{k}_{i14}^* = -\mathbf{k}_{i34}^*$  hold.

## 4.4 Derivation of mass matrices

As expressed in the derivation of FEM, the kinetic energy can be expressed as:

$$T_e = \frac{1}{2} \int_0^l \rho A \dot{u}^2 dx = \frac{1}{2} \dot{\mathbf{u}}^t \int_0^l \rho \mathbf{A} \mathbf{N}^t \mathbf{N} dx \dot{\mathbf{u}} \quad (4.32)$$

Where  $\rho A$  is the system parameter that is going to be replaced by the usual KL expansion:

$$\mathbf{m} = \int_{x=0}^l \mathbf{N}^t(\rho A) \mathbf{N} dx \quad (4.33)$$

$$\mathbf{m} = \int_{x=0}^l \mathbf{N}^t \left( \rho_0 A_0 + \sum_{i=1}^{n/2} (\lambda_i f_i(x) \xi_i + \lambda_i^* f_i^*(x) \xi_i^*) \right) \mathbf{N} dx \quad (4.34)$$

$$\mathbf{m} = \mathbf{m}_0 + \sum_{i=0}^{n/2} (\xi_i \lambda_i \mathbf{m}_i + \xi_i^* \lambda_i^* \mathbf{m}_i^*) \quad (4.35)$$

Matrices  $\mathbf{m}_0$ ,  $\mathbf{m}_i$  and  $\mathbf{m}_i^*$  are symmetric, and each of them has 16 terms: for each of them only 10 terms will have to be calculated.

$$\mathbf{m}_0 = \int_0^l \mathbf{N}^t(\rho A) \mathbf{N} dx \quad (4.36)$$

$$\mathbf{m}_0 = \frac{\rho_0 A_0 l}{420} \begin{bmatrix} 156 & & & & & \\ 22l & 4l^2 & \text{symmetric} & & & \\ 54 & 13l & 156 & & & \\ -13l & -3l^2 & -22l & 4l^2 & & \end{bmatrix} \quad (4.37)$$

For integrating  $\mathbf{m}_i$  and  $\mathbf{m}_i^*$ , we observe that the coordinates for the eigenfunctions are linked to the whole structure, and not to each element, whereas functions in  $\mathbf{N}$  are linked to each element. Integration will be done in element coordinates, and variables of eigenfunctions are going to be changed: if  $X$  is the coordinate used by the eigenfunction, and  $x$  the one used in a particular element,  $X = x + x_1^e$ , with  $x_1^e$  being the coordinate of the initial node of the element, related to the whole structure. The integrals giving the expression for each element of the mass matrix are

$$\begin{aligned} \mathbf{m}_{i11} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \right)^2 dx \\ \mathbf{m}_{i21} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \right) \left( x - \frac{2x^2}{l} + \frac{x^3}{l^2} \right) dx \\ \mathbf{m}_{i31} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \right) \left( \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right) dx \\ \mathbf{m}_{i41} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \right) \left( -\frac{x^2}{l} + \frac{x^3}{l^2} \right) dx \\ \mathbf{m}_{i22} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( x - \frac{2x^2}{l} + \frac{x^3}{l^2} \right)^2 dx \\ \mathbf{m}_{i32} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left( \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \right) \left( x - \frac{2x^2}{l} + \frac{x^3}{l^2} \right) dx \end{aligned}$$

$$\begin{aligned}
\mathbf{m}_{i_{42}} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) dx \\
\mathbf{m}_{i_{33}} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right)^2 dx \\
\mathbf{m}_{i_{43}} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) dx \\
\mathbf{m}_{i_{44}} &= \int_0^l \frac{\cos(\omega_n(x + x_1^e))}{\sqrt{a + \frac{\sin(2\omega_n a)}{2\omega_n}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right)^2 dx \\
\mathbf{m}_{i_{11}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right)^2 dx \\
\mathbf{m}_{i_{21}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) dx \\
\mathbf{m}_{i_{31}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) dx \\
\mathbf{m}_{i_{41}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right) \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) dx \\
\mathbf{m}_{i_{22}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right)^2 dx \\
\mathbf{m}_{i_{32}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) dx \\
\mathbf{m}_{i_{42}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \left(x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right) dx \\
\mathbf{m}_{i_{33}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right)^2 dx \\
\mathbf{m}_{i_{43}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right) \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right) dx \\
\mathbf{m}_{i_{44}^*} &= \int_0^l \frac{\sin(\omega_n(x + x_1^e))}{\sqrt{a - \frac{\sin(2\omega_n^* a)}{2\omega_n^*}}} \left(-\frac{x^2}{l} + \frac{x^3}{l^2}\right)^2 dx
\end{aligned}$$

The solutions of the integrals is not expressed explicitly: expressions become complicated and long, and can be obtained easily with any symbolic calculation program once the process for obtaining them is understood. This work has evidently been carried out and its application is exposed in the following section.



## 4.5 Numerical illustrations

The example beam model verifies that its boundary conditions are the ones of a clamped-free beam: zero vertical displacement and rotation on the left end. Heuristically, it is considered that this boundary condition is the one that will more likely cause a bigger variance in the response vector than the one obtained with other typical boundary conditions like clamped-clamped beam. The material properties are  $\rho = 7800 \text{ kg/m}^3$ ,  $E = 200 \text{ GPa}$ , values corresponding to steel. Geometrical variables values are: length of the beam  $L = 1.65 \text{ m}$  and rectangular cross section with length  $b = 40.06 \text{ mm}$  and width  $2.05 \text{ mm}$ . The variance of the parameters considered as random is 10% the value of the parameter:  $\sigma_\beta = 0.1 \beta$ . The parameters considered are  $\beta = AI_z$  for stiffness matrix and  $\beta = \rho A$  for mass matrix. The system response is calculated using equation (1.62): equation for a viscously damped proportional model where damping factor is  $\zeta_j = 0.015$ . The structure has 90 elements: when comparing at least the first 50 modes of the deterministic FEM with the analytical response obtained with equation

$$\eta_1 = 1.875 \quad \eta_2 = 4.694 \quad \eta_3 = 7.855 \quad \eta_{k>3} = (2k - 1) \frac{\pi}{2} \quad (4.38)$$

$$\omega_k = \sqrt{\frac{\eta_k^4 EI_z}{L^4 \rho A}} \quad (4.39)$$

the error between modes (analytical value minus numerical value divided by analytical value) is less than 1%. The 50-th natural frequency value is  $4.2364 \text{ Hz}$ , and the maximum frequency for which the structure is going to be tested is  $1.5 \text{ KHz}$  (the 30-th natural frequency).

Measure of the response vector is given for the forcing point, situated at the 3rd node, and at a sensor situated at the 27-th node.

The same four different correlation lengths as the ones used in the rod case are selected to research the effect of this value on the response statistics. We recall that those correlation lengths are  $0.1 L$ ,  $0.2 L$ ,  $0.5 L$  and  $L$ , where  $L$  is the beam total length. The uncorrelated random variables case is also studied. The number of terms in each KL expansion is chosen such that the ratio between the square roots of the first term and last term eigenvalues is less than 0.05, this leads to the same number of terms as in the former case, exposed in Table 3.1.

In Figure 4.1 and Figure 4.2 we show the mean of the dynamic response at two points on the bar. From these figures it is observed that the mean of the absolute value of the response does not coincide with the response of the deterministic system with mean values of stochastic parameters. Also, the larger is the correlation length, the more different are the mean responses from the deterministic response. This observation can be done for both sensors where the system response is collected. The larger the correlation length is, the more "damped" are the mean of the responses at high natural frequencies.

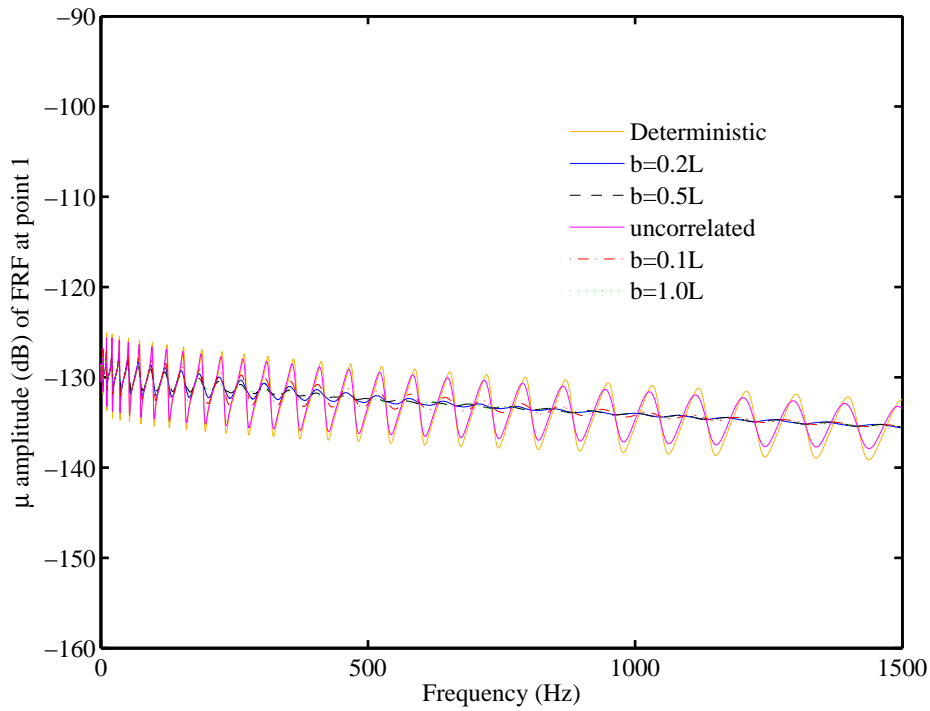


Figure 4.1: Absolute value of the FRF for the mean of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1, free end) .

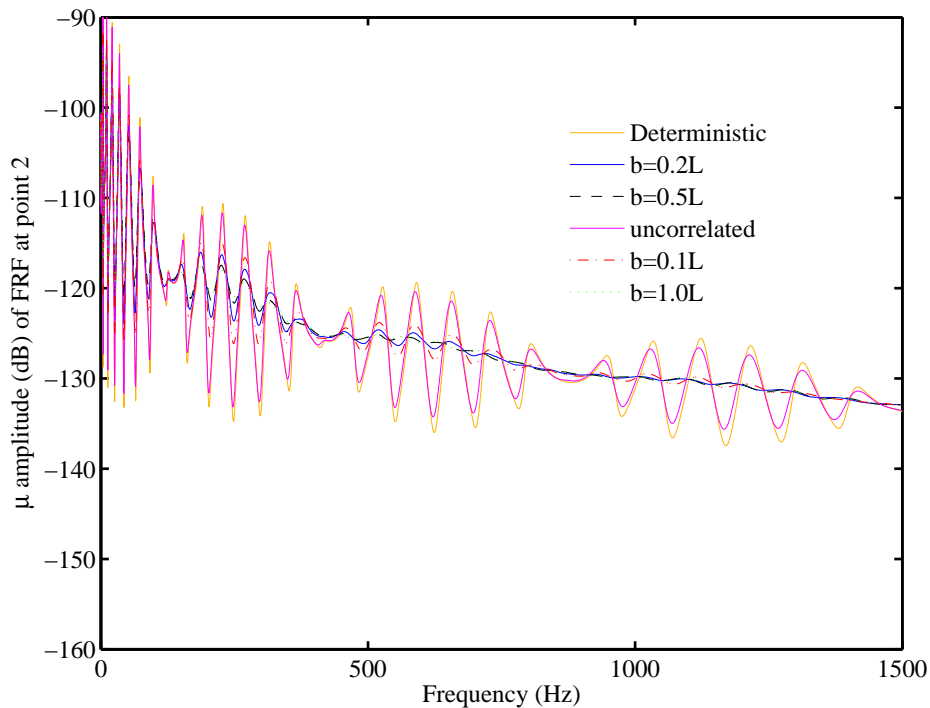


Figure 4.2: Absolute value of the FRF for the mean of the Monte Carlo simulations done directly with FEM for the sensor (Point 2, mid point).

In Figure 4.3 and Figure 4.5 we show the standard deviation of the dynamic response at two points on the bar. A significant impact of the correlation of the random field can be

observed on the dynamic response.

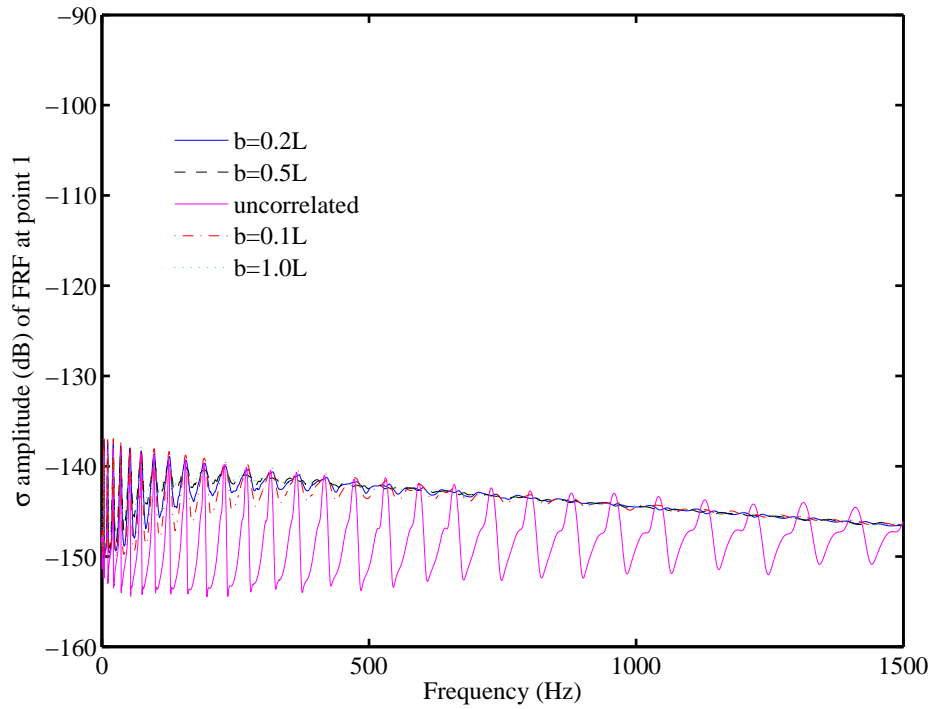


Figure 4.3: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1, free end).

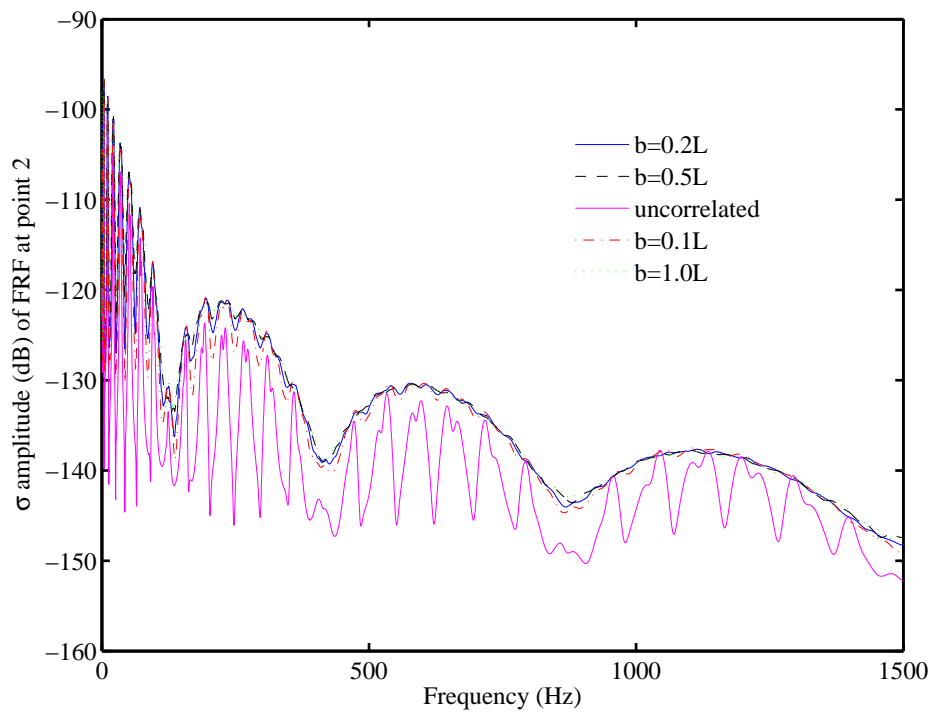


Figure 4.4: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the sensor (Point 2, mid point).

# Chapter 5

## Flexural vibration of thin plates with stochastic properties

### 5.1 Equation of motion

A thin plate of constant thickness  $h = 2b$  is subjected to distributed surface loads normal to the middle surface. It is assumed that the direct stress in the transverse direction,  $\sigma_z$  is zero and that normals to the middle surface of the undeformed plate remain straight and normal to the middle surface during deformation. Displacements parallel to the undeformed middle surface are given by:

$$u(x, y, z) = -z \frac{\partial w}{\partial x} \quad (5.1)$$

$$v(x, y, z) = -z \frac{\partial w}{\partial y} \quad (5.2)$$

Where  $w(x, y)$  is the displacement of the middle surface in the  $z$  direction. The strains are given by:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} \quad (5.3)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} \quad (5.4)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y} \quad (5.5)$$

$$(5.6)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad (5.7)$$

or, if  $\{\chi\}$  are the pseudo-strains, equations (5.3) to (5.5) can be rewritten in a vectorial way

$$\{\varepsilon\} = -z\{\chi\}. \quad (5.8)$$

Knowing that  $\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\}$ , with  $[\mathbf{D}]$  a symmetric matrix depending on the material, the strain energy for the element is

$$\begin{aligned}
 U &= \frac{1}{2} \int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) dV = \frac{1}{2} \int_V \{\boldsymbol{\sigma}\}^T \{\boldsymbol{\varepsilon}\} dV \\
 &= \frac{1}{2} \int_V \{\boldsymbol{\varepsilon}\}^T [\mathbf{D}] \{\boldsymbol{\varepsilon}\} dV \\
 U &= \frac{1}{2} \int_A \frac{h^3}{12} \{\boldsymbol{\chi}\}^T [\mathbf{D}] \{\boldsymbol{\chi}\} dA \tag{5.9} \\
 \mathbf{D} &= \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}
 \end{aligned}$$

If the rotary inertia terms are neglected, the kinetic energy of the plate and virtual work of the transverse loading are given by:

$$T = \frac{1}{2} \int_A \rho h \dot{w}^2 dA \tag{5.10}$$

$$\delta W = \int_A p_z \delta w dA \tag{5.11}$$

The bending moments per unit length in the plate are calculated:

$$\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_0^h z \boldsymbol{\alpha} dz = \int_0^h z^2 \mathbf{D} \boldsymbol{\chi} dz = -D \begin{bmatrix} \left[ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] \\ \left[ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] \\ (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{5.12}$$

The moments  $M_n$ ,  $M_s$  and  $M_{ns}$  are respectively the bending moment in plane  $Onz$  oriented along the tangent to the contour, in the perpendicular direction and the twisting moment about the normal. They are obtained from  $M_x, M_y$  and  $M_{xy}$  using equilibrium for a volume element on the plate boundary. And after introducing  $U$ ,  $T$  and  $W$  into Hamilton's Principle equation, we obtain the equation of motion, verified at any point on the surface.

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - m\ddot{w} + p = 0 \quad \text{where} \quad \begin{cases} Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \\ Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \end{cases} \tag{5.13}$$

while at any point on the contour, the boundary conditions must be satisfied:

$$\begin{cases} K_n = 0 & \text{or} & w = 0 \\ M_n = 0 & \text{or} & \frac{\partial w}{\partial n} = 0 \end{cases} \tag{5.14}$$

with  $K_n = \frac{\partial M_n}{\partial n} + 2 \frac{\partial M_{ns}}{\partial s} + \frac{1}{R}(M_n - M_s)$  the *Kirchhoff shear force*, and  $R$  the edge radius of curvature.

## 5.2 Deterministic Finite Element model

The measures of strain in plate bending analysis are the bending and twisting curvatures. Any displacement field has to verify that a rigid-body motion is possible, in the plate, this implies a translation in the  $z$  direction with  $w = \text{constant}$  and a rigid body rotation about the  $x$  and  $y$  axis of the form  $w = \text{constant}.y$  and  $w = \text{constant}.x$  respectively. It also verifies that a state of constant strain is possible, or, for a plate, a displacement with constant second derivatives of  $w$ ; and that the expression or deflection is smoothly continuous over the element. Another condition to verify is that of the continuity of derivatives, at interelement boundaries, of one order less than the highest-order derivative appearing in the potential energy expression: here, continuity of  $\partial w/\partial x$  and  $\partial w/\partial y$ . Then, for a 12-degree of freedom rectangular element, a deflection equation that verifies those conditions is

$$w = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{12} \end{bmatrix} \quad (5.15)$$

Each one of the four nodes of the element has three degrees of freedom  $w_i \phi_i \theta_i$  with  $\phi_i = \frac{\partial w}{\partial y}$  the rotation about the  $x$  axis and  $\theta_i = \frac{\partial w}{\partial x}$  the one about  $y$  axis for node  $i$ . As for rod and beam, it can be stated:

$$w = \alpha \mathbf{A} = \alpha \mathbf{C}^{-1} \mathbf{d} \quad (5.16)$$

With

$$\mathbf{C}^{-1} = \frac{1}{8a^3b^3} \begin{bmatrix} 2a^3b^3 & a^3b^4 & a^4b^3 & 2a^3b^3 & a^3b^4 & -b^3a^4 & 2a^3b^3 & -a^3b^4 & -a^4b^3 & 2a^3b^3 & -a^3b^4 & a^4b^3 \\ -3a^2b^3 & -a^2b^4 & -a^3b^3 & 3a^2b^3 & a^2b^4 & -a^3b^3 & 3a^2b^3 & -a^2b^4 & -a^3b^3 & -3a^2b^3 & a^2b^4 & -a^3b^3 \\ -3a^3b^2 & -a^3b^3 & -a^4b^2 & -3a^3b^2 & -a^3b^3 & a^4b^2 & 3a^3b^2 & -a^3b^3 & -a^4b^2 & 3a^3b^2 & -a^3b^3 & a^4b^2 \\ 0 & 0 & -a^2b^3 & 0 & 0 & a^2b^3 & 0 & 0 & a^2b^3 & 0 & 0 & -a^2b^3 \\ 4a^2b^2 & a^2b^3 & a^3b^2 & -4a^2b^2 & -a^2b^3 & a^3b^2 & 4a^2b^2 & -a^2b^3 & -a^3b^2 & -4a^2b^2 & a^2b^3 & -a^3b^2 \\ 0 & -a^3b^2 & 0 & 0 & -a^3b^2 & 0 & 0 & a^3b^2 & 0 & 0 & a^3b^2 & 0 \\ b^3 & 0 & ab^3 & -b^3 & 0 & ab^3 & -b^3 & 0 & ab^3 & b^3 & 0 & ab^3 \\ 0 & 0 & a^2b^2 & 0 & 0 & -a^2b^2 & 0 & 0 & a^2b^2 & 0 & 0 & -a^2b^2 \\ 0 & a^2b^2 & 0 & 0 & -a^2b^2 & 0 & 0 & a^2b^2 & 0 & 0 & -a^2b^2 & 0 \\ a^3 & a^3b & 0 & a^3 & a^3b & 0 & -a^3 & a^3b & 0 & -a^3 & a^3b & 0 \\ -b^2 & 0 & -ab^2 & b^2 & 0 & -ab^2 & -b^2 & 0 & ab^2 & b^2 & 0 & ab^2 \\ -a^2 & -a^2b & 0 & a^2 & a^2b & 0 & -a^2 & a^2b & 0 & a^2 & -a^2b & 0 \end{bmatrix}$$

As for rod and beam, a stiffness, mass and forcing vector can be obtained from the expressions of strain and kinetic energy, and virtual work. The strain energy is:

$$U_p = \frac{1}{2} \iint \boldsymbol{\varepsilon}_p^t \mathbf{E}_p \boldsymbol{\varepsilon}_p dx dy = \frac{1}{2} \mathbf{d}^t \mathbf{K} \mathbf{d} \quad \mathbf{E}_p = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (5.17)$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (5.18)$$



with

$$\mathbf{m}_e = 4ab \begin{bmatrix} 1 & 0 & 0 & \frac{a^2}{3} & 0 & \frac{b^2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a^2}{3} & 0 & 0 & 0 & 0 & \frac{a^4}{5} & 0 & \frac{a^2b^2}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{b^2}{3} & 0 & 0 & 0 & 0 & \frac{a^2b^2}{9} & 0 & \frac{b^4}{5} & 0 & 0 \\ \frac{a^2}{3} & 0 & 0 & \frac{a^4}{5} & 0 & \frac{a^2b^2}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a^2b^2}{9} & 0 & 0 & 0 & 0 & 0 & \frac{a^4b^2}{15} & \frac{a^2b^4}{15} \\ \frac{b^2}{3} & 0 & 0 & \frac{a^2b^2}{9} & 0 & \frac{b^4}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{a^4}{5} & 0 & 0 & 0 & 0 & \frac{a^6}{7} & 0 & \frac{a^4b^2}{15} & 0 & 0 & 0 \\ 0 & 0 & \frac{a^2b^2}{9} & 0 & 0 & 0 & 0 & \frac{a^4b^2}{15} & 0 & \frac{a^2b^4}{15} & 0 & 0 \\ 0 & \frac{a^2b^2}{9} & 0 & 0 & 0 & 0 & \frac{a^4b^2}{15} & 0 & \frac{a^2b^4}{15} & 0 & 0 & 0 \\ 0 & 0 & \frac{b^4}{5} & 0 & 0 & 0 & 0 & \frac{a^2b^4}{15} & 0 & \frac{b^6}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{a^4b^2}{15} & 0 & 0 & 0 & 0 & 0 & \frac{a^6b^2}{21} & \frac{a^4b^4}{25} \\ 0 & 0 & 0 & 0 & \frac{a^2b^4}{15} & 0 & 0 & 0 & 0 & 0 & \frac{a^4b^4}{25} & \frac{a^2b^6}{21} \end{bmatrix}$$

Now, the KL expansion of parameters appearing in deterministic mass and stiffness matrices expressions allows the calculation of matrices used for the calculation of the dispersion parameter.

### 5.3 Derivation of stiffness and mass matrices

As explained, assumptions are introduced (like Kirchhoff hypothesis) in order to convert the equations of three-dimensional elasticity into a theoretical (but approximate) model of plate behavior in which deformation throughout the plate is expressed solely in terms of the middle surface deflection  $w$ . Classical plate behavior is characterized by the variable  $w$ : the governing differential equations of equilibrium, when written in terms of displacements, are now of fourth order. If the pseudo-strains are

$$\boldsymbol{\chi}_p = - \{ \partial^2 w / \partial x^2 \quad \partial^2 w / \partial y^2 \quad 2\partial^2 w / \partial x \partial y \} \quad (5.22)$$

the strain energy is given by:

$$U_p = \frac{1}{2} \iint \boldsymbol{\chi}_p^t \mathbf{E}_p \boldsymbol{\chi}_p dx dy = \frac{1}{2} \mathbf{d}^t \mathbf{K}_e \mathbf{d} \quad (5.23)$$

with

$$\mathbf{K}_e = \int_{A_e} \mathbf{B}_p^t \mathbf{E}_p \mathbf{B}_p dA_e = (\mathbf{C}^{-1})^t \left( \int_{A_e} \boldsymbol{\beta}_p^t \mathbf{E}_p \boldsymbol{\beta}_p dA_e \right) \mathbf{C}^{-1} = \mathbf{C}^{-t} \mathbf{k}_e \mathbf{C}^{-1} \quad (5.24)$$

and  $\mathbf{d}$  the vector of displacements at each node,  $\mathbf{K}_e$  the stiffness matrix and  $D$  given by (5.18). The kinetic energy is given by:

$$T_p = \frac{1}{2} \dot{\mathbf{d}}^t \int \rho h \mathbf{C}^{-t} \boldsymbol{\alpha}^t \boldsymbol{\alpha} \mathbf{C} dA_e \dot{\mathbf{d}} = \frac{1}{2} \dot{\mathbf{d}}^t \mathbf{M}_e \dot{\mathbf{d}} \quad (5.25)$$



$$\mathbf{M}_e = \mathbf{C}^{-t} \int \rho h \boldsymbol{\alpha}^t \boldsymbol{\alpha} dA_e \mathbf{C} = \mathbf{C}^{-t} \mathbf{m}_e \mathbf{C} \quad (5.26)$$

with  $\mathbf{d}$  the vector of displacements at each node and  $\mathbf{M}_e$  the mass matrix of the element.

Variables  $D$ , appearing in the expression of  $\mathbf{E}_p$ , and  $m = \rho h$  are going to be expanded with a KL expansion, assuming a correlation function

$$C(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2) = e^{-|\mathbf{x}_1 - \mathbf{x}_2|c_1 - |\mathbf{y}_1 - \mathbf{y}_2|c_2} \quad (5.27)$$

Following the Kl expansion in subsection 1.1.4, we consider  $n_1$  number of terms in the  $x$ -direction and  $n_2$  number of terms in the  $y$ -direction. Combining we have

$$D = D_0 + \sum_{i=1}^{n_1/2} \sum_{j=1}^{n_2/2} (\xi_{ij} \lambda_{ij} f_{ij} + \xi_{i^*j} \lambda_{i^*j} f_{i^*j} + \xi_{ij^*} \lambda_{ij^*} f_{ij^*} + \xi_{i^*j^*} \lambda_{i^*j^*} f_{i^*j^*}) \quad (5.28)$$

$$m = m_0 + \sum_{i=1}^{n_1/2} \sum_{j=1}^{n_2/2} (\xi_{ij} \lambda_{ij} f_{ij} + \xi_{i^*j} \lambda_{i^*j} f_{i^*j} + \xi_{ij^*} \lambda_{ij^*} f_{ij^*} + \xi_{i^*j^*} \lambda_{i^*j^*} f_{i^*j^*}) \quad (5.29)$$

with

$$\lambda_{ij} = \frac{4c_1 c_2}{(\omega_{iX}^2 + c_1)(\omega_{jY}^2 + c_2)} \quad (5.30)$$

$$f_{ij} = \frac{\cos(w_{iX})}{\sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} \frac{\cos(w_{jY})}{\sqrt{b + \frac{\sin(2\omega_j b)}{2\omega_j}}} \quad (5.31)$$

$$\lambda_{i^*j} = \frac{4c_1 c_2}{(\omega_{iX}^{*2} + c_1)(\omega_{jY}^2 + c_2)} \quad (5.32)$$

$$f_{i^*j} = \frac{\sin(w_{iX}^*)}{\sqrt{a - \frac{\sin(2\omega_i^* a)}{2\omega_i^*}}} \frac{\cos(w_{jY})}{\sqrt{b + \frac{\sin(2\omega_j b)}{2\omega_j}}} \quad (5.33)$$

$$\lambda_{ij^*} = \frac{4c_1 c_2}{(\omega_{iX}^2 + c_1)(\omega_{jY}^{*2} + c_2)} \quad (5.34)$$

$$f_{ij^*} = \frac{\cos(w_{iX})}{\sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} \frac{\sin(w_{jY}^*)}{\sqrt{b - \frac{\sin(2\omega_j^* b)}{2\omega_j^*}}} \quad (5.35)$$

$$\lambda_{i^*j^*} = \frac{4c_1 c_2}{(\omega_{iX}^{*2} + c_1)(\omega_{jY}^{*2} + c_2)} \quad (5.36)$$

$$f_{i^*j^*} = \frac{\sin(w_{iX}^*)}{\sqrt{a - \frac{\sin(2\omega_i^* a)}{2\omega_i^*}}} \frac{\sin(w_{jY}^*)}{\sqrt{b - \frac{\sin(2\omega_j^* b)}{2\omega_j^*}}} \quad (5.37)$$

For all those equations, coordinates  $x$  and  $y$  are given with respect to some global axes placed in a corner of the plate. Those coordinates do not coincide with the coordinates of each element of the FEM  $x$  and  $y$ , their relation with a given element coordinates is

$$x = x_o^e + x \quad \text{and} \quad y = y_o^e + y \quad (5.38)$$



or

$$I_n(z) = \int z^m \sin(wz + a) dz \quad \text{with} \quad m = 1, \dots, 6 \quad (5.43)$$

The constant  $w$  is  $w_i$ ,  $w_i^*$ ,  $w_j$  or  $w_j^*$  and  $a$  is a constant equal to the product  $a = wz_o^e$  ( $z_o^e$  can be  $x_o^e$  for  $z = x$  or  $y_o^e$  for  $z = y$ ). A more detailed description on the derivation of the terms of mass and stiffness matrices are given in Appendix A.

## 5.4 Numerical illustrations

The example model's boundary conditions are the ones of a clamped on one edge plate: zero vertical displacement and rotation along both edges at positions that verify  $x = 0$ , that is, one of the narrow edges of the plate. Heuristically, it is considered that this boundary condition is the one that will more likely cause a bigger variance in the response vector than the one obtained with other typical boundary conditions like displacement equal to zero everywhere on the contour (boundary condition where an analytical solution is available). Material properties are taken as  $\rho = 7860 \text{ kg/m}^3$ ,  $E = 200 \text{ GPa}$  and  $\nu = 0.3$ , values corresponding to steel. Geometrical variables are: length and width of the plate  $L = 0.998 \text{ m}$ ,  $W = 0.59 \text{ m}$  and thickness  $t = 3.0 \text{ mm}$ . The variance of the parameters considered as random is 10% the value of the parameter:  $\sigma_\beta = 0.1 \beta$ . The parameters considered are  $\beta = D = Et^3/(12(1-\nu^2))$  for stiffness matrix and  $\beta = \rho t$  for mass matrix. The system response is calculated using equation (1.62): equation for a viscously damped proportional model where damping factor is  $\zeta_j = 0.02$ . The structure has its length divided into 25 elements ( $x$  direction) and its width divided into 15 elements: so, the model has 375 elements. No analytical solution is available for these boundary conditions, so, it is not possible to compare the natural frequencies obtained from the numerical calculations with an analytical expression as it was done for rod and beam.

The maximum frequency for which the structure is going to be tested is  $4.0 \text{ KHz}$ . Measure of the response vector is given for the forcing point, situated at the 180-th node, that is, coordinates  $x = 4$  and  $y = 6$ , and at a sensor situated at the 307-th node (coordinates  $x = 7$  and  $y = 11$ ).

The same four different normalized correlation lengths as the ones used in the rod and beam cases are selected to research the effect of this value on the response statistics. We recall that the non-normalized correlation lengths are  $0.1d$ ,  $0.2d$ ,  $0.5d$  and  $d$ , where  $d$  is the plate dimension considered (width or length) As it has been said in the description of KL expansion for the plate case, each direction can have a different associated correlation length. Here the normalized correlation length linked to each direction, is, for each Monte Carlo Simulation: 0.1, 0.2, 0.5 and 1. The uncorrelated random variables case is also studied. The number of terms in each KL expansion is chosen such that the ratio between the square roots of the first term and last term eigenvalues is less than 0.05, verifying also that the KL expansion in both directions has the same number of terms. Now, each eigenvalue is going to

be the product of two eigenvalues, each one coming from one different 1D KL expansion. So, the last eigenvalue is the product of the two last eigenvalues of each 1D KL expansion, and so, the first eigenvalue is the product of the two first eigenvalues of each 1D KL expansion. In other words, The terms of each 1D KL expansion is given in Table 5.1 When simulating the

Correlation length/d	0.1	0.2	0.5	1
Terms of KL expansion	16	9	6	6

Table 5.1: Number of terms used in KL expansion for each normalized correlation length

system matrices, Wishart matrices are fitted following criteria 3 in section 2.3. The number of samples for Monte Carlo simulations are 1000 both for the FE model and Random Matrix direct Monte Carlo simulation. Each sample of the direct Monte Carlo simulation of FEM is done as a deterministic case where the variables are correlated following the correlation function already described in KL expansion.

In Figure 5.1 and Figure 5.2 we show the mean of the dynamic response at two points on the bar. From these figures it is observed that the mean of the absolute value of the response does not coincide with the response of the deterministic system with mean values of stochastic parameters. Also, the larger is the correlation length, the more different are the mean responses from the deterministic response. This observation can be done for both sensors where the system response is collected. The larger the correlation length is, the more "damped" are the mean of the responses at high natural frequencies.

In Figure 5.3 and Figure 5.4 we show the standard deviation of the dynamic response at two points on the bar. It is noticed that the impact of correlation of variables in the system response variance for direct Monte Carlo simulations of FEM is not as big as the one observed for rod and beam models.

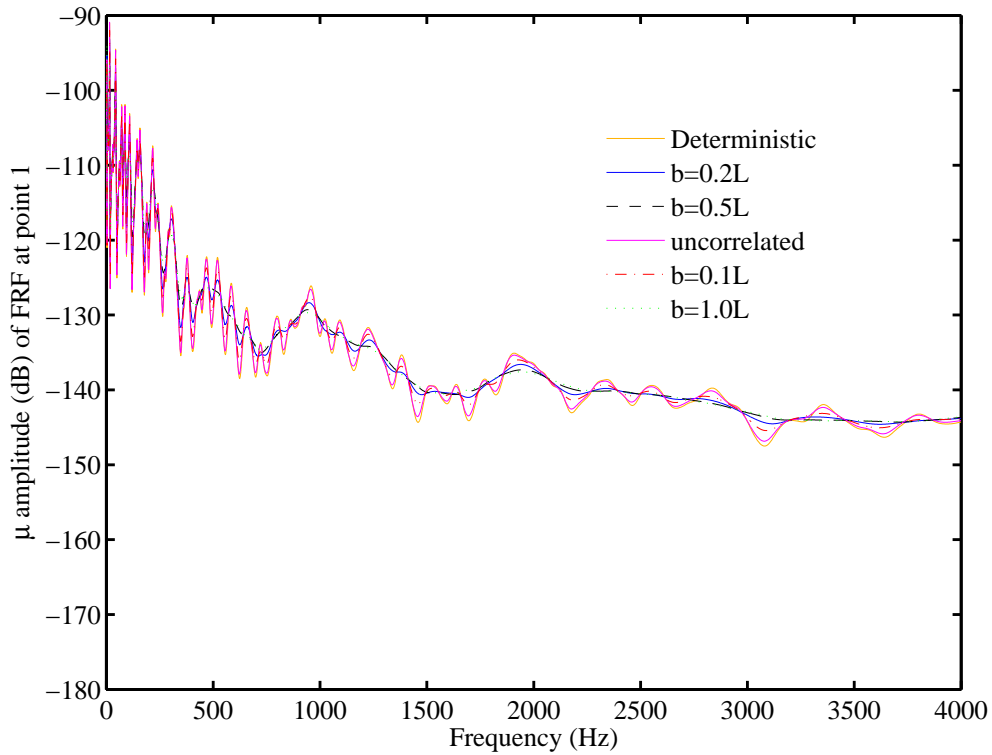


Figure 5.1: Absolute value of the FRF for the mean of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1) .

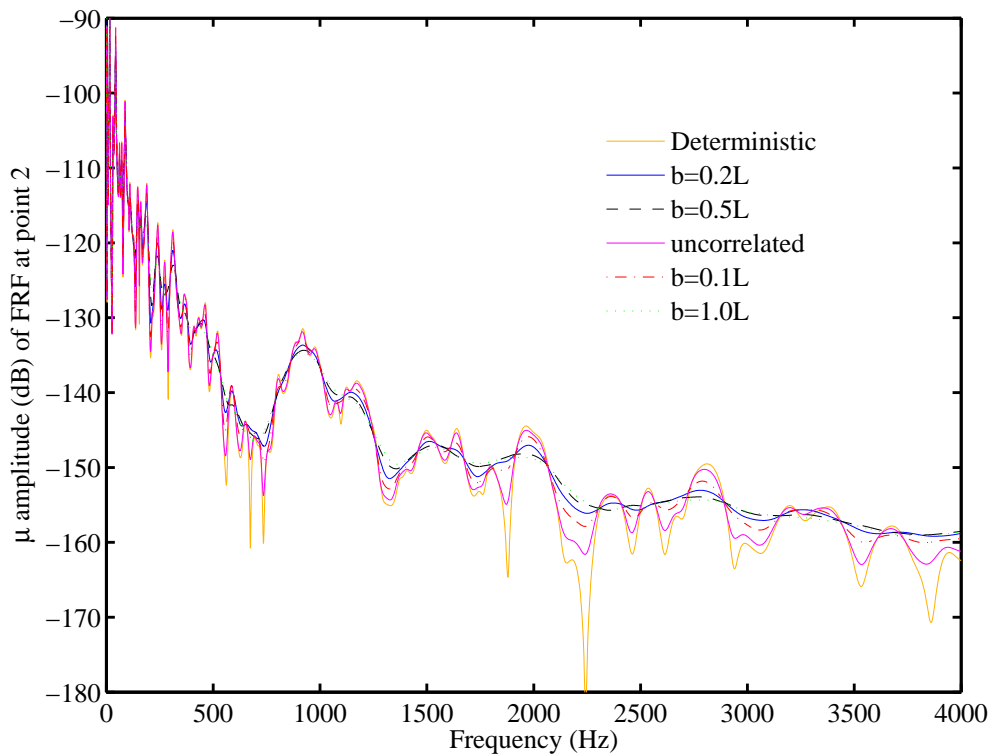


Figure 5.2: Absolute value of the FRF for the mean of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1) and the sensor (Point 2).

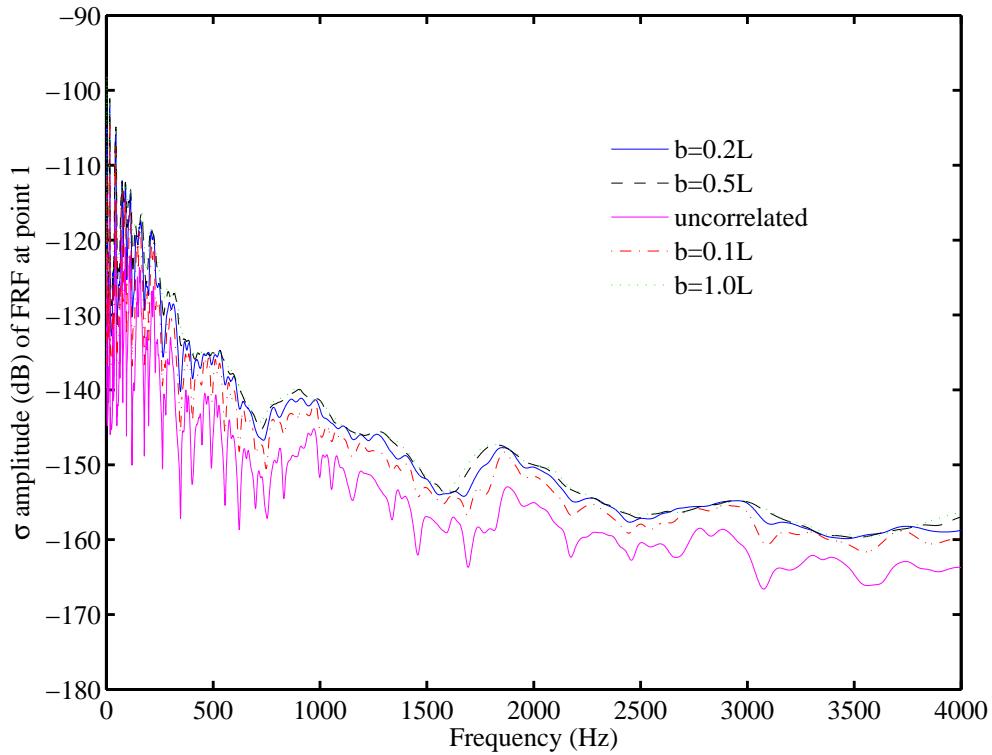


Figure 5.3: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the excitation point (Point 1).

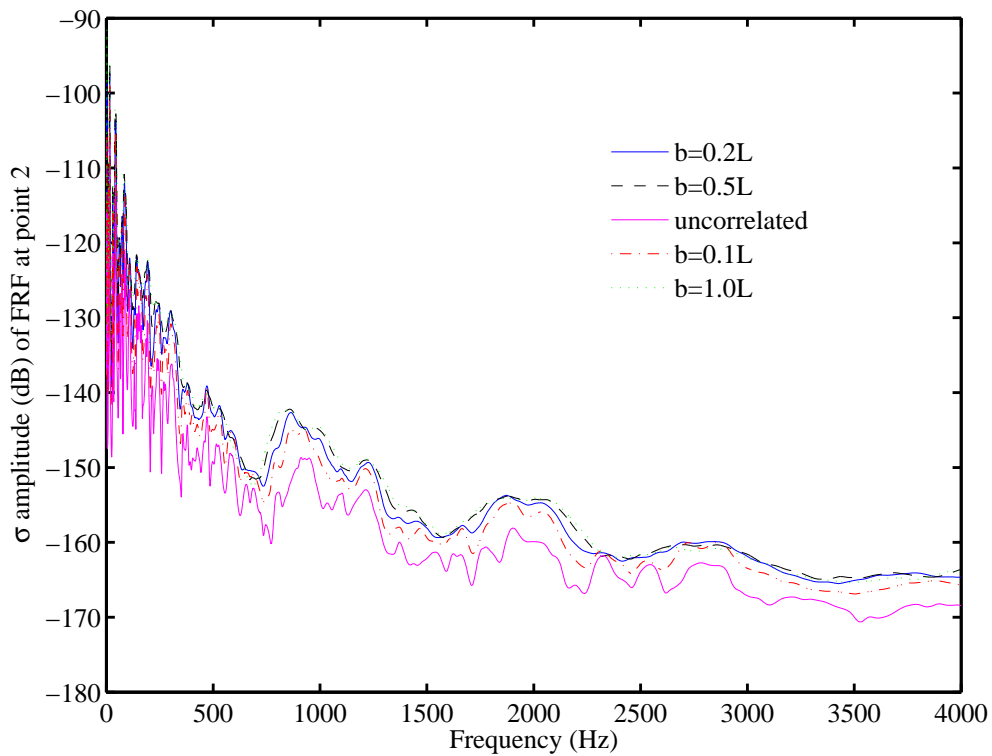


Figure 5.4: Absolute value of the FRF for the standard deviation of the Monte Carlo simulations done directly with FEM for the sensor (Point 2).

# Appendix A

## The elements of the stochastic mass and stiffness matrices of a plate

In Chapter 4 was explained how to derive the stiffness and mass matrices, for a plate, using Karhunen-Loève expansion. Each of the elements of those matrices for each eigenfunction  $f$ , are:

$$\mathbf{k}_{e44} = \int_{-a}^a \int_{-b}^b 4 f dx dy$$

$$\mathbf{k}_{e64} = \int_{-a}^a \int_{-b}^b 4\nu f dx dy$$

$$\mathbf{k}_{e74} = \int_{-a}^a \int_{-b}^b 12x f dx dy$$

$$\mathbf{k}_{e77} = \int_{-a}^a \int_{-b}^b 36x^2 f dx dy$$

$$\mathbf{k}_{e85} = \int_{-a}^a \int_{-b}^b 4x(1 - \nu) f dx dy$$

$$\mathbf{k}_{e87} = \int_{-a}^a \int_{-b}^b 12xy f dx dy$$

$$\mathbf{k}_{e94} = \int_{-a}^a \int_{-b}^b 4x\nu f dx dy$$

$$\mathbf{k}_{e96} = \int_{-a}^a \int_{-b}^b 4x f dx dy$$

$$\mathbf{k}_{e98} = \int_{-a}^a \int_{-b}^b 4xy(2 - \nu) f dx dy$$

$$\mathbf{k}_{e104} = \int_{-a}^a \int_{-b}^b 12y\nu f dx dy$$

$$\mathbf{k}_{e107} = \int_{-a}^a \int_{-b}^b 36xy\nu f dx dy$$

$$\mathbf{k}_{e109} = \int_{-a}^a \int_{-b}^b 12xy f dx dy$$

$$\mathbf{k}_{e55} = \int_{-a}^a \int_{-b}^b (2 - 2\nu) f dx dy$$

$$\mathbf{k}_{e66} = \int_{-a}^a \int_{-b}^b 4 f dx dy$$

$$\mathbf{k}_{e76} = \int_{-a}^a \int_{-b}^b 12x\nu f dx dy$$

$$\mathbf{k}_{e84} = \int_{-a}^a \int_{-b}^b 4y f dx dy$$

$$\mathbf{k}_{e86} = \int_{-a}^a \int_{-b}^b 4y\nu f dx dy$$

$$\mathbf{k}_{e88} = \int_{-a}^a \int_{-b}^b (4y^2 + 8x^2(1 - \nu)) f dx dy$$

$$\mathbf{k}_{e95} = \int_{-a}^a \int_{-b}^b 4y(1 - \nu) f dx dy$$

$$\mathbf{k}_{e97} = \int_{-a}^a \int_{-b}^b 12x^2\nu f dx dy$$

$$\mathbf{k}_{e99} = \int_{-a}^a \int_{-b}^b 4(x^2 + 2y^2(1 - \nu)) f dx dy$$

$$\mathbf{k}_{e106} = \int_{-a}^a \int_{-b}^b 12y f dx dy$$

$$\mathbf{k}_{e108} = \int_{-a}^a \int_{-b}^b 12y^2\nu f dx dy$$

$$\mathbf{k}_{e1010} = \int_{-a}^a \int_{-b}^b 36y^2 f dx dy$$

$$\begin{aligned}
\mathbf{k}_{e_{114}} &= \int_{-a}^a \int_{-b}^b 12xy f dx dy & \mathbf{k}_{e_{115}} &= \int_{-a}^a \int_{-b}^b 6x^2(1-\nu) f dx dy \\
\mathbf{k}_{e_{116}} &= \int_{-a}^a \int_{-b}^b 12xy\nu f dx dy & \mathbf{k}_{e_{117}} &= \int_{-a}^a \int_{-b}^b 36x^2y f dx dy \\
\mathbf{k}_{e_{118}} &= \int_{-a}^a \int_{-b}^b 12x(y^2 + x^2(1-\nu)) f dx dy & \mathbf{k}_{e_{119}} &= \int_{-a}^a \int_{-b}^b 12x^2y f dx dy f dx dy \\
\mathbf{k}_{e_{1110}} &= \int_{-a}^a \int_{-b}^b 36xy^2\nu f dx dy & \mathbf{k}_{e_{1111}} &= \int_{-a}^a \int_{-b}^b 18x^2(2y^2 + x^2(1-\nu)) f dx dy \\
\mathbf{k}_{e_{124}} &= \int_{-a}^a \int_{-b}^b 12xy\nu f dx dy & \mathbf{k}_{e_{125}} &= \int_{-a}^a \int_{-b}^b 6y^2(1-\nu) f dx dy \\
\mathbf{k}_{e_{126}} &= \int_{-a}^a \int_{-b}^b 12xy f dx dy & \mathbf{k}_{e_{127}} &= \int_{-a}^a \int_{-b}^b 36x^2y\nu f dx dy \\
\mathbf{k}_{e_{128}} &= \int_{-a}^a \int_{-b}^b 12xy^2 f dx dy & \mathbf{k}_{e_{129}} &= \int_{-a}^a \int_{-b}^b 12y(x^2 + y^2(1-\nu)) f dx dy \\
\mathbf{k}_{e_{1210}} &= \int_{-a}^a \int_{-b}^b 36xy^2 f dx dy & \mathbf{k}_{e_{1211}} &= \int_{-a}^a \int_{-b}^b 18x^2y^2(1+\nu) f dx dy \\
\mathbf{k}_{e_{1212}} &= \int_{-a}^a \int_{-b}^b 18y^2(2x^2 + y^2(1-\nu)) f dx dy
\end{aligned}$$

$$\begin{aligned}
\mathbf{m}_{e_{11}} &= \int_{-a}^a \int_{-b}^b f dx dy & \mathbf{m}_{e_{21}} &= \int_{-a}^a \int_{-b}^b x f dx dy & \mathbf{m}_{e_{22}} &= \int_{-a}^a \int_{-b}^b x^2 f dx dy \\
\mathbf{m}_{e_{31}} &= \int_{-a}^a \int_{-b}^b y f dx dy & \mathbf{m}_{e_{32}} &= \int_{-a}^a \int_{-b}^b xy f dx dy & \mathbf{m}_{e_{33}} &= \int_{-a}^a \int_{-b}^b y^2 f dx dy \\
\mathbf{m}_{e_{41}} &= \int_{-a}^a \int_{-b}^b x^2 f dx dy & \mathbf{m}_{e_{42}} &= \int_{-a}^a \int_{-b}^b x^3 f dx dy & \mathbf{m}_{e_{43}} &= \int_{-a}^a \int_{-b}^b x^2y f dx dy \\
\mathbf{m}_{e_{44}} &= \int_{-a}^a \int_{-b}^b x^4 f dx dy & \mathbf{m}_{e_{51}} &= \int_{-a}^a \int_{-b}^b xyf dx dy & \mathbf{m}_{e_{52}} &= \int_{-a}^a \int_{-b}^b x^2yf dx dy \\
\mathbf{m}_{e_{53}} &= \int_{-a}^a \int_{-b}^b xy^2f dx dy & \mathbf{m}_{e_{54}} &= \int_{-a}^a \int_{-b}^b x^3yf dx dy & \mathbf{m}_{e_{55}} &= \int_{-a}^a \int_{-b}^b x^2y^2f dx dy \\
\mathbf{m}_{e_{61}} &= \int_{-a}^a \int_{-b}^b y^2f dx dy & \mathbf{m}_{e_{62}} &= \int_{-a}^a \int_{-b}^b xy^2f dx dy & \mathbf{m}_{e_{63}} &= \int_{-a}^a \int_{-b}^b y^3f dx dy \\
\mathbf{m}_{e_{64}} &= \int_{-a}^a \int_{-b}^b x^2y^2f dx dy & \mathbf{m}_{e_{65}} &= \int_{-a}^a \int_{-b}^b xy^3f dx dy & \mathbf{m}_{e_{66}} &= \int_{-a}^a \int_{-b}^b y^4f dx dy \\
\mathbf{m}_{e_{71}} &= \int_{-a}^a \int_{-b}^b x^3f dx dy & \mathbf{m}_{e_{72}} &= \int_{-a}^a \int_{-b}^b x^4f dx dy & \mathbf{m}_{e_{73}} &= \int_{-a}^a \int_{-b}^b x^3yf dx dy \\
\mathbf{m}_{e_{74}} &= \int_{-a}^a \int_{-b}^b x^5f dx dy & \mathbf{m}_{e_{75}} &= \int_{-a}^a \int_{-b}^b x^4yf dx dy & \mathbf{m}_{e_{76}} &= \int_{-a}^a \int_{-b}^b x^3y^2f dx dy \\
\mathbf{m}_{e_{77}} &= \int_{-a}^a \int_{-b}^b x^6f dx dy & \mathbf{m}_{e_{81}} &= \int_{-a}^a \int_{-b}^b x^2yf dx dy & \mathbf{m}_{e_{82}} &= \int_{-a}^a \int_{-b}^b x^3yf dx dy
\end{aligned}$$



$$\begin{array}{lll}
\mathbf{m}_{e83} = \int_{-a}^a \int_{-b}^b x^2 y^2 f \, dx \, dy & \mathbf{m}_{e84} = \int_{-a}^a \int_{-b}^b x^4 y f \, dx \, dy & \mathbf{m}_{e85} = \int_{-a}^a \int_{-b}^b x^3 y^2 f \, dx \, dy \\
\mathbf{m}_{e86} = \int_{-a}^a \int_{-b}^b x^2 y^3 f \, dx \, dy & \mathbf{m}_{e87} = \int_{-a}^a \int_{-b}^b x^5 y f \, dx \, dy & \mathbf{m}_{e88} = \int_{-a}^a \int_{-b}^b x^4 y^2 f \, dx \, dy \\
\mathbf{m}_{e91} = \int_{-a}^a \int_{-b}^b x y^2 f \, dx \, dy & \mathbf{m}_{e92} = \int_{-a}^a \int_{-b}^b x^2 y^2 f \, dx \, dy & \mathbf{m}_{e93} = \int_{-a}^a \int_{-b}^b x y^3 f \, dx \, dy \\
\mathbf{m}_{e94} = \int_{-a}^a \int_{-b}^b x^3 y^2 f \, dx \, dy & \mathbf{m}_{e95} = \int_{-a}^a \int_{-b}^b x^2 y^3 f \, dx \, dy & \mathbf{m}_{e96} = \int_{-a}^a \int_{-b}^b y^4 x f \, dx \, dy \\
\mathbf{m}_{e97} = \int_{-a}^a \int_{-b}^b x^4 y^2 f \, dx \, dy & \mathbf{m}_{e98} = \int_{-a}^a \int_{-b}^b x^3 y^3 f \, dx \, dy & \mathbf{m}_{e99} = \int_{-a}^a \int_{-b}^b x^2 y^4 f \, dx \, dy \\
\mathbf{m}_{e101} = \int_{-a}^a \int_{-b}^b y^3 f \, dx \, dy & \mathbf{m}_{e102} = \int_{-a}^a \int_{-b}^b x y^3 f \, dx \, dy & \mathbf{m}_{e103} = \int_{-a}^a \int_{-b}^b y^4 f \, dx \, dy \\
\mathbf{m}_{e104} = \int_{-a}^a \int_{-b}^b x^2 y^3 f \, dx \, dy & \mathbf{m}_{e105} = \int_{-a}^a \int_{-b}^b y^4 x f \, dx \, dy & \mathbf{m}_{e106} = \int_{-a}^a \int_{-b}^b y^5 f \, dx \, dy \\
\mathbf{m}_{e107} = \int_{-a}^a \int_{-b}^b x^3 y^3 f \, dx \, dy & \mathbf{m}_{e108} = \int_{-a}^a \int_{-b}^b x^2 y^4 f \, dx \, dy & \mathbf{m}_{e109} = \int_{-a}^a \int_{-b}^b y^5 x f \, dx \, dy \\
\mathbf{m}_{e110} = \int_{-a}^a \int_{-b}^b y^6 f \, dx \, dy & \mathbf{m}_{e111} = \int_{-a}^a \int_{-b}^b x^3 y f \, dx \, dy & \mathbf{m}_{e112} = \int_{-a}^a \int_{-b}^b x^4 y f \, dx \, dy \\
\mathbf{m}_{e113} = \int_{-a}^a \int_{-b}^b x^3 y^2 f \, dx \, dy & \mathbf{m}_{e114} = \int_{-a}^a \int_{-b}^b x^5 y f \, dx \, dy & \mathbf{m}_{e115} = \int_{-a}^a \int_{-b}^b x^4 y^2 f \, dx \, dy \\
\mathbf{m}_{e116} = \int_{-a}^a \int_{-b}^b x^3 y^3 f \, dx \, dy & \mathbf{m}_{e117} = \int_{-a}^a \int_{-b}^b x^6 y f \, dx \, dy & \mathbf{m}_{e118} = \int_{-a}^a \int_{-b}^b x^5 y^2 f \, dx \, dy \\
\mathbf{m}_{e119} = \int_{-a}^a \int_{-b}^b x^4 y^3 f \, dx \, dy & \mathbf{m}_{e1110} = \int_{-a}^a \int_{-b}^b x^3 y^4 f \, dx \, dy & \mathbf{m}_{e1111} = \int_{-a}^a \int_{-b}^b x^6 y^2 f \, dx \, dy \\
\mathbf{m}_{e121} = \int_{-a}^a \int_{-b}^b x y^3 f \, dx \, dy & \mathbf{m}_{e122} = \int_{-a}^a \int_{-b}^b x^2 y^3 f \, dx \, dy & \mathbf{m}_{e123} = \int_{-a}^a \int_{-b}^b y^4 x f \, dx \, dy \\
\mathbf{m}_{e124} = \int_{-a}^a \int_{-b}^b x^3 y^3 f \, dx \, dy & \mathbf{m}_{e125} = \int_{-a}^a \int_{-b}^b x^2 y^4 f \, dx \, dy & \mathbf{m}_{e126} = \int_{-a}^a \int_{-b}^b y^5 x f \, dx \, dy \\
\mathbf{m}_{e127} = \int_{-a}^a \int_{-b}^b x^4 y^3 f \, dx \, dy & \mathbf{m}_{e128} = \int_{-a}^a \int_{-b}^b x^3 y^4 f \, dx \, dy & \mathbf{m}_{e129} = \int_{-a}^a \int_{-b}^b x^2 y^5 f \, dx \, dy \\
\mathbf{m}_{e1210} = \int_{-a}^a \int_{-b}^b y^6 x f \, dx \, dy & \mathbf{m}_{e1211} = \int_{-a}^a \int_{-b}^b x^4 y^4 f \, dx \, dy & \mathbf{m}_{e1212} = \int_{-a}^a \int_{-b}^b x^2 y^6 f \, dx \, dy
\end{array}$$

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