Day 2: Mechanics of irregular cellular viscoelastic metamaterials

Sondipon Adhikari, FRAeS

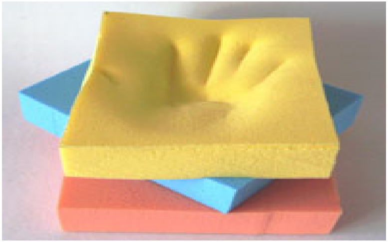
Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Bay Campus, Swansea, Wales, UK, Email: S.Adhikari@swansea.ac.uk
Twitter: @ProfAdhikari, Web: http://engweb.swan.ac.uk/~adhikaris

SPARC Course: Metamaterial and metasandwich for energy harvesting and vibration control

March 27, 2021

Outline

Examples of some viscoelastic materials



(a) Viscoelastic foam

(b) Viscoelastic membrane

(c) Viscoelastic sheet

(d) Viscoelastic sheet

Fundamental equation for the viscoelastic behaviour

- When a linear viscoelastic model is employed, the stress at some point of a structure can be expressed as a convolution integral over a kernel function as
$$\sigma(t) = \int_{-\infty}^{t} g(t-\tau)\frac{\partial \epsilon(\tau)}{\partial \tau}\,\tau \tag{1}$$
- $t \in \mathbb{R}^+$ is the time, $\sigma(t)$ is stress and $\epsilon(t)$ is strain.
- The kernel function $g(t)$ also known as 'hereditary function', 'relaxation function' or 'after-effect function' in the context of different subjects.
- In practice, the kernel function is often defined in the frequency domain (or Laplace domain). Taking the Laplace transform of Equation (1), we have
$$\bar{\sigma}(s) = s\bar{G}(s)\bar{\epsilon}(s) \tag{2}$$
Here $\bar{\sigma}(s)$, $\bar{\epsilon}(s)$ and $\bar{G}(s)$ are Laplace transforms of $\sigma(t)$, $\epsilon(t)$ and $g(t)$ respectively and $s \in \mathbb{C}$ is the (complex) Laplace domain parameter.

Mathematical representation of the kernel function

- The kernel function in Equation (2) is a complex function in the frequency domain. For notational convenience we denote

$$\bar{G}(s) = \bar{G}(\mathrm{i}\omega) = G(\omega) \tag{3}$$

  where $\omega \in \mathbb{R}^+$ is the frequency.

- The complex modulus $G(\omega)$ can be expressed in terms of its real and imaginary parts or in terms of its amplitude and phase as follows

$$G(\omega) = G'(\omega) + \mathrm{i}G''(\omega) = |G(\omega)|e^{\mathrm{i}\phi(\omega)} \tag{4}$$

  The real and imaginary parts of the complex modulus, that is, $G'(\omega)$ and $G''(\omega)$ are also known as the storage and loss moduli respectively.

- One of the main restriction on the form of the kernel function comes from the fact that the response of the structure must not start before the application of the forces.
- This causality condition imposes a mathematical relationship between real and imaginary parts of the complex modulus, known as Kramers-Kronig relations

Mathematical representation of the kernel function

- Kramers-Kronig relations specifies that the real and imaginary parts should be related by a Hilbert transform pair, but can be general otherwise. Mathematically this can be expressed as

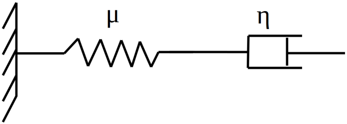
$$\begin{aligned} G'(\omega) &= G_\infty + \frac{2}{\pi}\int_0^\infty \frac{uG''(u)}{\omega^2 - u^2}\mathrm{d}u \\ G''(\omega) &= \frac{2\omega}{\pi}\int_0^\infty \frac{G'(u)}{u^2 - \omega^2}\mathrm{d}u \end{aligned} \tag{5}$$

  where the unrelaxed modulus $G_\infty = G(\omega \to \infty) \in \mathbb{R}$.
- Equivalent relationships linking the modulus and the phase of $G(\omega)$ can expressed as

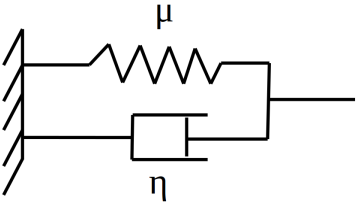
$$\begin{aligned} \ln|G'(\omega)| &= \ln|G_\infty| + \frac{2}{\pi}\int_0^\infty \frac{u\phi(u)}{\omega^2 - u^2}\mathrm{d}u \\ \phi(\omega) &= \frac{2\omega}{\pi}\int_0^\infty \frac{\ln|G(u)|}{u^2 - \omega^2}\mathrm{d}u \end{aligned} \tag{6}$$

- Complex modulus derived using a physics based principle automatically satisfy these conditions. However, there can be many other function which would also satisfy these condition.

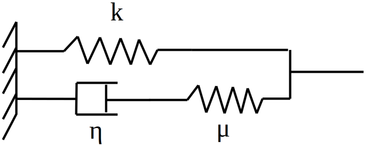
Viscoelastic models



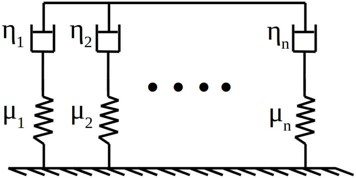
(e) Maxwell model



(f) Voigt model



(g) Standard linear model



(h) Generalised Maxwell model

Figure: Springs and dashpots based models viscoelastic materials.

Viscoelastic models

The viscoelastic kernel function can be expressed for the four models as

- *Maxwell model:*

$$g(t) = \mu e^{-(\mu/\eta)t}\mathcal{U}(t) \tag{7}$$

- *Voigt model:*

$$g(t) = \eta\delta(t) + \mu\mathcal{U}(t) \tag{8}$$

- *Standard linear model:*

$$g(t) = E_R\left[1 - (1 - \frac{\tau_\sigma}{\tau_\epsilon})e^{-t/\tau_\epsilon}\right]\mathcal{U}(t) \tag{9}$$

- *Generalised Maxwell model:*

$$g(t) = \left[\sum_{j=1}^{n} \mu_j e^{-(\mu_j/\eta_j)t}\right]\mathcal{U}(t) \tag{10}$$

Models similar to this is also known as the Pony series model.

Viscoelastic models

Viscoelastic model	Complex modulus
Biot model	$G(\omega) = G_0 + \sum_{k=1}^{n} \frac{a_k \mathrm{i}\omega}{\mathrm{i}\omega + b_k}$
Fractional derivative	$G(\omega) = \frac{G_0 + G_\infty (\mathrm{i}\omega\tau)^\beta}{1 + (\mathrm{i}\omega\tau)^\beta}$
GHM	$G(\omega) = G_0 \left[1 + \sum_k \alpha_k \frac{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega}{-\omega^2 + 2\mathrm{i}\xi_k\omega_k\omega + \omega_k^2}\right]$
ADF	$G(\omega) = G_0 \left[1 + \sum_{k=1}^{n} \Delta_k \frac{\omega^2 + \mathrm{i}\omega\Omega_k}{\omega^2 + \Omega_k^2}\right]$
Step-function	$G(\omega) = G_0 \left[1 + \eta \frac{1 - e^{-st_0}}{st_0}\right]$
Half cosine model	$G(\omega) = G_0 \left[1 + \eta \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}\right]$
Gaussian model	$G(\omega) = G_0 \left[1 + \eta\, e^{\omega^2/4\mu} \left\{1 - \mathrm{erf}\left(\frac{\mathrm{i}\omega}{2\sqrt{\mu}}\right)\right\}\right]$

Complex modulus for some viscoelastic models in the frequency domain

The Biot Model

- We consider that each constitutive element of a hexagonal unit within the lattice structure is modelled using viscoelastic properties. For simplicity, we use Biot model with only one term. Frequency dependent complex elastic modulus for an element is expresses as

$$E(\omega) = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \tag{11}$$

  where μ and ϵ are the relaxation parameter and a constant defining the 'strength' of viscosity, respectively. E_s is the intrinsic Young's modulus.

- The amplitude of this complex elastic modulus is given by

$$|E(\omega)| = E_S \sqrt{\frac{\mu^2 + \omega^2 (1+\epsilon)^2}{\mu^2 + \omega^2}} \tag{12}$$

- The phase (ϕ) of this complex elastic modulus is given by

$$\phi\big(E(\omega)\big) = \tan^{-1}\left(\frac{\epsilon\mu\omega}{\mu^2 + \omega^2(1+\epsilon)}\right) \tag{13}$$

Irregular lattice structures

- The equivalent elastic properties for a regular lattice:

$$E_1 = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{14}$$

$$E_2 = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{15}$$

$$\nu_{12} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta} \tag{16}$$

$$\nu_{21} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{17}$$

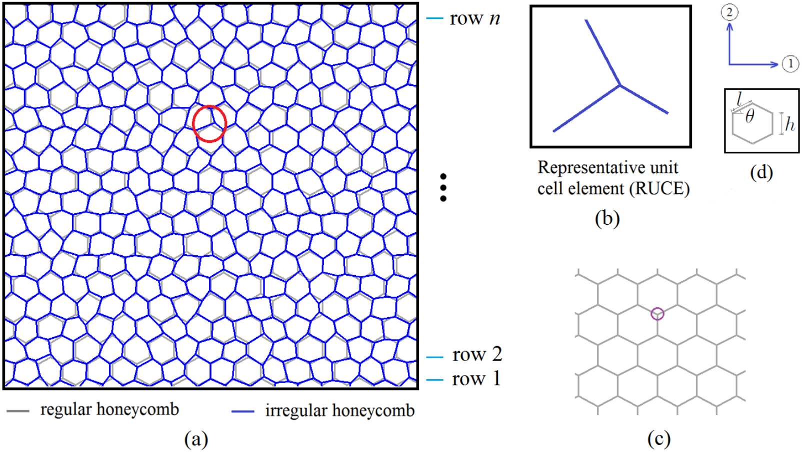
$$G_{12} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{18}$$

- Parameters of these expressions CANNOT be randomised to obtain equivalent properties for an irregular lattice.

Irregular lattice structures

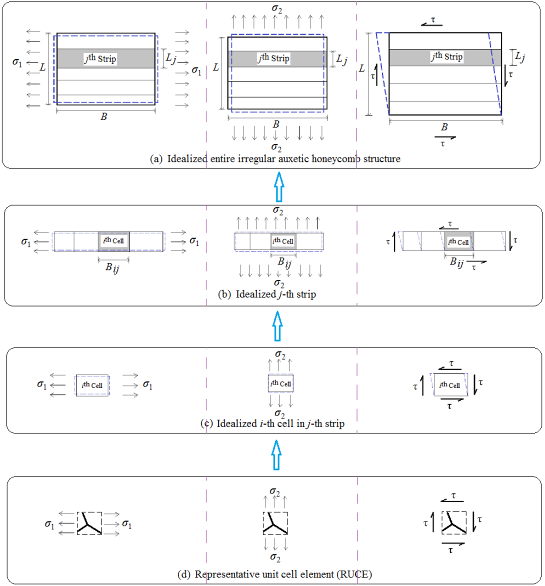
- Direct numerical simulation to deal with irregularity in lattice structures may not necessarily provide proper understanding of the underlying physics of the system. An analytical approach could be a simple, insightful, yet an efficient way to obtain effective elastic properties of lattice structures.
- This work develops a structural mechanics based analytical framework for predicting equivalent in-plane elastic properties of irregular lattices having spatially random variations in cell angles.
- Closed-form analytical expressions will be derived for equivalent in-plane elastic properties.
- An approach based on the frequency-domain representation of the viscoelastic property of the constituent elements in the cells is used.

The philosophy of the analytical approach for irregular lattices

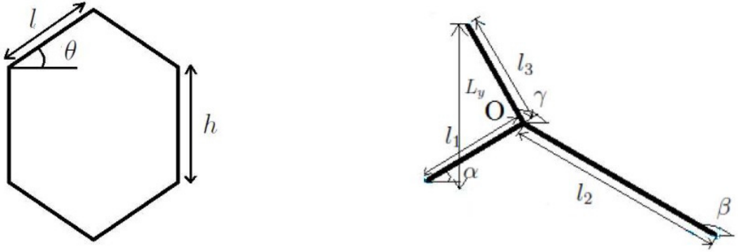


(a) Typical representation of an irregular lattice (b) Representative unit cell element (RUCE) (c) Illustration to define degree of irregularity (d) Unit cell considered for regular hexagonal lattice by Gibson and Ashby (1999).

The idealisation of RUCE and the bottom-up homogenisation approach

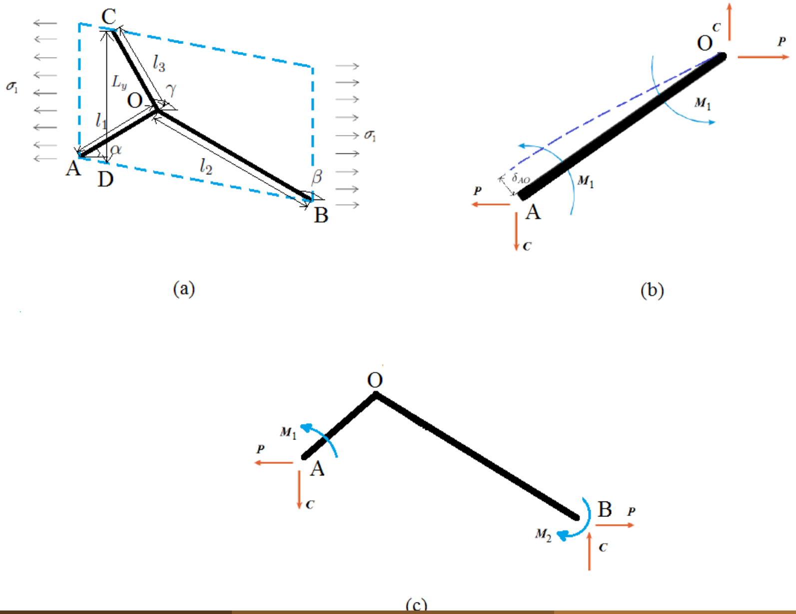


Unit cell geometry



(a) Classical unit cell for regular lattices (b) Representative unit cell element (RUCE) geometry for irregular lattices

RUCE and free-body diagram for the derivation of E_1



Longitudinal Young's modulus for an idealized RUCE

- Stress σ_1 is applied in direction-1 for deriving the expression of longitudinal Young's modulus for a single RUCE (E_{1U}). From the condition of vertical equilibrium, it can be concluded that the vertical forces acting on points A and B should be of equal magnitude and opposite sign.
- The horizontal forces acting on points A and B can be expressed as $P = \sigma_1 L_y b$, where L_y represents the length CD and b is the height of honeycomb sheet (dimension perpendicular to the 1-2 plane).
- The moments M_1 and M_2 can be expressed as

$$M_1 = \frac{1}{2}(P l_1 \sin\alpha - C l_1 \cos\alpha) \tag{19}$$

$$M_2 = \frac{1}{2}(P l_2 \sin\beta - C l_2 \cos\beta) \tag{20}$$

- Considering the rotational equilibrium of the free-body diagram presented in, the expression for C can be obtained as

$$C = P\left(\frac{l_1 \sin\alpha - l_2 \sin\beta}{l_1 \cos\alpha - l_2 \cos\beta}\right) \tag{21}$$

Longitudinal Young's modulus for an idealized RUCE

- The horizontal deflection of point A with respect to point O (δ^h_{AO}) consists of the deflection due to force P and the force C

$$\delta^h_{AO} = \left(\frac{P l_1^3 \sin\alpha}{12 E_s I} - \frac{C l_1^3 \cos\alpha}{12 E_s I} \right) \sin\alpha \tag{22}$$

  where the first and second terms in the bracket represents the deflection of point A with respect to point O in the direction perpendicular to AO due to forces P and C respectively.
- The superscript h is used to represent horizontal direction of the applied stress. Here, E_s represents the intrinsic material property of the material, by which the honeycomb cell walls (/connecting members) are made of.
- The notation I represents the second moment of area of the cell walls, i.e. $I = bt^3/12$, where t denotes the thickness of honeycomb cell wall.

Longitudinal Young's modulus for an idealized RUCE

- The horizontal deflection of point B with respect to point O can be expressed as

$$\delta_{BO}^{h} = \left(\frac{P l_2^3 \sin\beta}{12 E_s I} - \frac{C l_1^3 \cos\beta}{12 E_s I} \right) \sin\beta \tag{23}$$

- The distance of the point vertically below joint O and on the line AB is given by

$$\delta_{O} = \frac{l_2 \sin\beta l_1 \cos\alpha - l_1 \sin\alpha l_2 \cos\beta}{l_1 \cos\alpha - l_2 \cos\beta} \tag{24}$$

Considering a linear strain field along the line AB, the effective horizontal deformation of the RUCE is given by

$$\begin{aligned} \delta_1^h &= \delta_{AO}^h \frac{\delta_O}{l_1 \sin\alpha} + \delta_{BO}^h \frac{\delta_O}{l_2 \sin\beta} \\ &= \frac{\sigma_1 L_y l_1^2 l_2^2 \left(l_1 + l_2\right) \left(\cos\alpha \sin\beta - \sin\alpha \cos\beta\right)^2}{E_s t^3 \left(l_1 \cos\alpha - l_2 \cos\beta\right)^2} \end{aligned} \tag{25}$$

Longitudinal Young's modulus for an idealized RUCE

- The strain in direction-1 can be obtained from 25 as

$$\epsilon_1^h = \frac{\sigma_1 L_y l_1^2 l_2^2 \left(l_1 + l_2\right) \left(\cos\alpha \sin\beta - \sin\alpha \cos\beta\right)^2}{E_s t^3 \left(l_1 \cos\alpha - l_2 \cos\beta\right)^3} \tag{26}$$

From 26, elastic modulus of a single RUCE in direction-1 is expressed as

$$E_{1U} = \frac{E_s t^3 \left(l_1 \cos\alpha - l_2 \cos\beta\right)^3}{L_y l_1^2 l_2^2 \left(l_1 + l_2\right) \left(\cos\alpha \sin\beta - \sin\alpha \cos\beta\right)^2} \tag{27}$$

Longitudinal Young's modulus for a Non-idealized RUCE

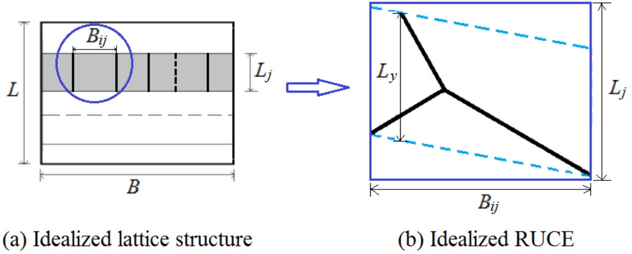


Figure: Idealization scheme of RUCE and the irregular lattice structure

- The expression of E_{1U} is for a non-idealized RUCE having a dimension of L_y in direction-2. However, for assembling the local properties of RUCEs conveniently to the global level, it is essential to obtain the equivalent material property of an idealized RUCE (E_{1U}^{I}) that has a virtual dimension of L_j (dimension of the j^{th} strip in direction-2).

Longitudinal Young's modulus for a Non-idealized RUCE

- Considering a linear strain field, E_{1U}^{I} can be obtained based on the deformation compatibility condition along direction-1, i. e. the deformation of the idealized RUCE and non-idealized RUCE in direction-1 should be equal

$$\frac{PB_{ij}}{A_{NI}E_{1U}} = \frac{PB_{ij}}{A_I E_{1U}^{I}} \tag{28}$$

Here $A_{NI} = L_y b$ and $A_I = L_j b$. The above equation can be reduced to

$$E_{1U}^{I} = E_{1U}\frac{L_y}{L_j} \tag{29}$$

Transverse Young's modulus of the entire irregular lattice

- The deformation compatibility of j^{th} strip ensures that the total deformation of the strip in direction-1 due to stress σ_1 (Δ_{1j}) is the summation of individual deformations in direction-1 of each idealized RUCE (Δ_{1ij}), while deformation of the idealized RUCEs of that strip in direction-2 are same. Thus for the j^{th} strip

$$\Delta_{1j} = \sum_{i=1}^{m} \Delta_{1ij} \tag{30}$$

- The 30 can be rewritten as

$$\epsilon_{1j} B_j = \sum_{i=1}^{m} \epsilon_{1ij} B_{ij} \tag{31}$$

where ϵ_{1j} and B_j represent total strain and dimension in direction-1 for the j^{th} strip. Here $B_{ij} = (l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij})$ and $B_j = \sum_{i=1}^{m} B_{ij}$.

Transverse Young's modulus of the entire irregular lattice

- Equation (31) leads to

$$\frac{\sigma_1 B_j}{\hat{E_{1j}}} = \sum_{i=1}^{m} \frac{\sigma_1 B_{ij}}{E^I_{1Uij}} \tag{32}$$

  From 32, equivalent Young's modulus of j^{th} strip ($\hat{E_{1j}}$) can be expressed as

$$\hat{E_{1j}} = \frac{B_j}{\sum\limits_{i=1}^{m} \frac{B_{ij}}{E^I_{1Uij}}} \tag{33}$$

  where E^I_{1Uij} is the equivalent longitudinal elastic modulus in direction-1 of a single idealized RUCE positioned at (i,j) that can be obtained from equation (29).

- In the next step, closed-form expression for equivalent longitudinal Young's modulus of the entire irregular lattice (E_{1eq}) is obtained using the equivalent longitudinal Young's modulus for a single strip ($\hat{E_{1j}}$).

Transverse Young's modulus of the entire irregular lattice

- Employing the force equilibrium conditions and deformation compatibility condition we have

$$\sigma_1 L b = \sum_{j=1}^{n} \sigma_{1j} L_j b \tag{34}$$

where L_j is the dimension of j^{th} strip in direction-2 and $L = \sum_{j=1}^{n} L_j$. The notation b represents the dimension of the lattice in the perpendicular direction to 1-2 plane.

- As strains in direction-1 for each of the n strips are the same to satisfy the deformation compatibility condition, equation (34) leads to

$$E_{1eq} L = \sum_{j=1}^{n} \hat{E_{1j}} L_j \tag{35}$$

Transverse Young's modulus of the entire irregular lattice

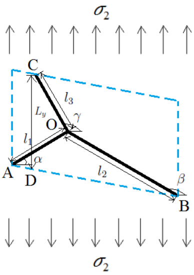
- Using 33 and 35, the equivalent Young's modulus in direction-1 of the entire irregular honeycomb structure (E_{1eq}) can be expressed as

$$E_{1eq} = \frac{1}{L}\sum_{j=1}^{n} \frac{B_j L_j}{\sum\limits_{i=1}^{m} \dfrac{B_{ij}}{E_{1Uij}^{I}}} \tag{36}$$

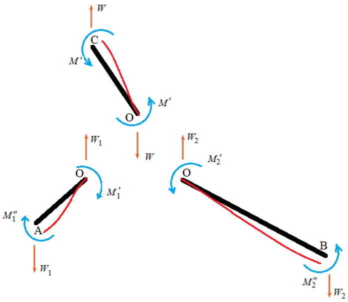
- From equations (27), (29) and (36), the expression for the longitudinal elastic modulus of the entire irregular lattice can be written as

$$E_{1eq} = \frac{E_s t^3}{L}\sum_{j=1}^{n} \frac{\sum\limits_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum\limits_{i=1}^{m} \dfrac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij}+l_{2ij}\right)\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)^2}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2}} \tag{37}$$

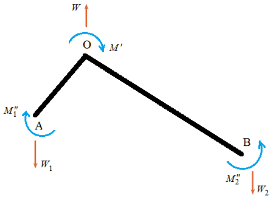
RUCE and free-body diagram for the derivation of E_2



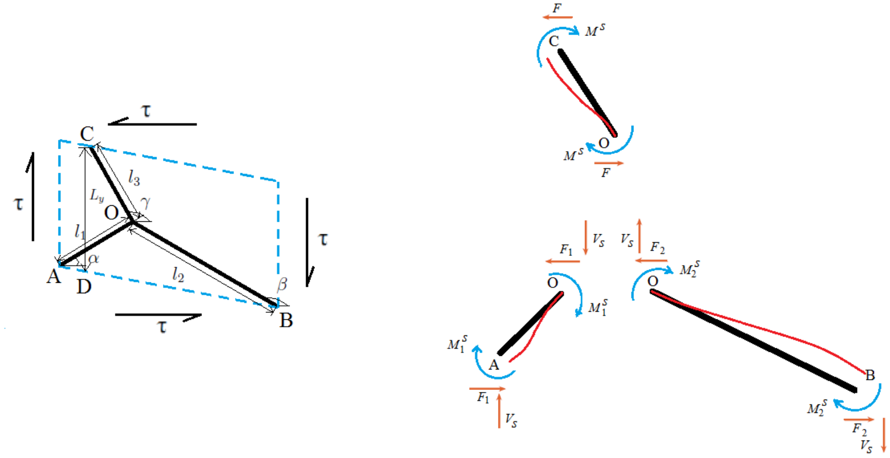
(a)



(b)



RUCE and free-body diagram for the derivation of G_{12}



(a) (b)

Equivalent E_1, E_2

Equivalent E_1

$$E_{1v}(\omega) = \frac{t^3}{L}\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)^2}{E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)\left(\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2\right)}} \tag{38}$$

Equivalent Young's moduli E_2

$$E_{2v}(\omega) = \frac{Lt^3}{\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)\left(l_{3ij}^2\cos^2\gamma_{ij}\left(l_{3ij}+\frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right)+\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos^2\alpha_{ij}\cos^2\beta_{ij}}{\left(l_{1ij}\cos\alpha_{ij}-l_{2ij}\cos\beta_{ij}\right)^2}\right)^{-1}}} \tag{39}$$

Equivalent shear Modulus G_{12}

Equivalent G_{12}

$$G_{12v}(\omega) = \frac{Lt^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} (l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij})}{\sum_{i=1}^{m} E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)\left(l_{3ij}^2 \sin^2\gamma_{ij}\left(l_{3ij}+\frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right)\right)^{-1}}} \quad (40)$$

Poisson's ratios ν_{12}, ν_{21}

Equivalent ν_{12}

$$\nu_{12eq} = -\frac{1}{L}\sum_{j=1}^{n} \frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\cos\alpha_{ij}\cos\beta_{ij}}} \tag{41}$$

Equivalent ν_{21}

$$\nu_{21eq} = -\frac{L}{\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos\alpha_{ij}\cos\beta_{ij}\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2\left(l_{3ij}^2\cos^2\gamma_{ij}\left(l_{3ij} + \frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right) + \frac{l_{1ij}^2 l_{2ij}^2 (l_{1ij}+l_{2ij})\cos^2\alpha_{ij}\cos^2\beta_{ij}}{(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij})^2}\right)}}} \tag{42}$$

Only spatial variation of the material properties

- According to the notations used for a regular lattice by Gibson and Ashby (1999), the notations for lattices without any structural irregularity can be expressed as: $L = n(h + l\sin\theta)$; $l_{1ij} = l_{2ij} = l_{3ij} = l$; $\alpha_{ij} = \theta$; $\beta_{ij} = 180^\circ - \theta$; $\gamma_{ij} = 90^\circ$, for all i and j.
- Using these transformations in case of the spatial variation of only material properties, the closed-form formulae for compound variation of material and geometric properties (equations 38–40) can be reduced to:

$$E_{1v} = \kappa_1 \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{43}$$

$$E_{2v} = \kappa_2 \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{44}$$

and

$$G_{12v} = \kappa_2 \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{45}$$

Only spatial variation of the material properties

- The multiplication factors κ_1 and κ_2 arising due to the consideration of spatially random variation of intrinsic material properties can be expressed as

$$\kappa_1 = \frac{m}{n} \sum_{j=1}^{n} \frac{1}{\sum_{i=1}^{m} \frac{1}{E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)}} \tag{46}$$

and

$$\kappa_2 = \frac{n}{m} \frac{1}{\sum_{j=1}^{n} \frac{1}{\sum_{i=1}^{m} E_{sij}\left(1+\epsilon_{ij}\frac{\mathrm{i}\omega}{\mu_{ij}+\mathrm{i}\omega}\right)}} \tag{47}$$

- In the special case when $\omega \to 0$ and there is no spatial variabilities in the material properties of the lattice, all viscoelastic material properties become identical (i.e. $E_{sij} = E_s$, $\mu_{ij} = \mu$ and $\epsilon_{ij} = \epsilon$ for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$) and subsequently the amplitude of κ_1 and κ_2 becomes exactly 1. This confirms that the expressions in 46 and 47 give the necessary generalisations of the classical expressions of Gibson and Ashby (1999) through 43–45.

Only geometric irregularities

- In case of only spatially random variation of structural geometry but constant viscoelastic material properties (i.e. $E_{sij} = E_s$, $\mu_{ij} = \mu$ and $\epsilon_{ij} = \epsilon$ for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$) the 38–40 lead to

$$E_{1v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega} \right) \zeta_1 \tag{48}$$

$$E_{2v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega} \right) \zeta_2 \tag{49}$$

$$G_{12v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega} \right) \zeta_3 \tag{50}$$

Only geometric irregularities

- The random coefficients $\zeta_i \quad (i = 1, 2, 3)$ are

$$\zeta_1 = \frac{t^3}{L} \sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij} + l_{2ij} \right) \left(\cos \alpha_{ij} \sin \beta_{ij} - \sin \alpha_{ij} \cos \beta_{ij} \right)^2}{\left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)^2}} \tag{51}$$

$$\zeta_2 = \frac{Lt^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} \left(l_{3ij}^2 \cos^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) + \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij} + l_{2ij} \right) \cos^2 \alpha_{ij} \cos^2 \beta_{ij}}{\left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)^2} \right)^{-1}}} \tag{52}$$

$$\zeta_3 = \frac{Lt^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos \alpha_{ij} - l_{2ij} \cos \beta_{ij} \right)}{\sum_{i=1}^{m} \left(l_{3ij}^2 \sin^2 \gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) \right)^{-1}}} \tag{53}$$

Regular hexagonal lattices

- The geometric notations for regular lattices can be expressed as: $L = n(h + l\sin\theta)$; $l_{1ij} = l_{2ij} = l_{3ij} = l$; $\alpha_{ij} = \theta$; $\beta_{ij} = 180^\circ - \theta$; $\gamma_{ij} = 90^\circ$, for all i and j. Using these transformations, the expressions of in-plane elastic moduli for regular hexagonal lattices (without the viscoelastic effect) can be obtained.
- The in-plane Young's moduli and shear modulus (viscosity dependent in-plane elastic properties) can be expressed as

$$E_{1v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{54}$$

$$E_{2v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{55}$$

$$G_{12v} = E_s\left(1 + \epsilon\frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right)\left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2(1 + 2\frac{h}{l})\cos\theta} \tag{56}$$

- The amplitude of the elastic moduli obtained based on the above expressions converge to the closed-form equation provided by Gibson and Ashby (1999) in the limiting case of $\omega \to 0$.

Regular uniform hexagonal lattices

- In the case of regular uniform lattices with $\theta = 30^\circ$, we have

$$E_{1v} = E_{2v} = 2.3 E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \tag{57}$$

- Similarly, in the case of shear modulus for regular uniform lattices $(\theta = 30^\circ)$

$$G_{12v} = 0.57 E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \tag{58}$$

- Regular viscoelastic lattices satisfy the reciprocal theorem

$$E_{2v}\nu_{12v} = E_{1v}\nu_{21v} = E_S \left(1 + \epsilon \frac{\mathrm{i}\omega}{\mu + \mathrm{i}\omega}\right) \left(\frac{t}{l}\right)^3 \frac{1}{\sin\theta\cos\theta} \tag{59}$$

Random field model for material and geometric properties

- Correlated structural and material attributes can be modelled random fields $\mathcal{H}(\mathbf{x}, \theta)$.
- The traditional way of dealing with random field is to discretise the random field into finite number of random variables. The available schemes for discretising random fields can be broadly divided into three groups: (1) point discretisation (e.g., midpoint method, shape function method, integration point method, optimal linear estimate method); (2) average discretisation method (e.g., spatial average, weighted integral method), and (3) series expansion method (e.g., orthogonal series expansion).
- An advantageous alternative for discretising $\mathcal{H}(\mathbf{x}, \theta)$ is to represent it in a generalised Fourier type of series as, often termed as Karhunen-Loève (KL) expansion.

Karhunen-Loève (KL) expansion

- Suppose, $\mathcal{H}(\mathbf{x}, \theta)$ is a random field with covariance function $\Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2)$ defined in the probability space $(\Theta, \mathcal{F}, \mathcal{P})$. The KL expansion for $\mathcal{H}(\mathbf{x}, \theta)$ takes the following form

$$\mathcal{H}(\mathbf{x}, \theta) = \bar{\mathcal{H}}(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \psi_i(\mathbf{x}) \tag{60}$$

  where $\{\xi_i(\theta)\}$ is a set of uncorrelated random variables.
- $\{\lambda_i\}$ and $\{\psi_i(\mathbf{x})\}$ are the eigenvalues and eigenfunctions of the covariance kernel $\Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2)$, satisfying the integral equation

$$\int_{\Re^N} \Gamma_{\mathcal{H}}(\mathbf{x}_1, \mathbf{x}_2) \psi_i(\mathbf{x}_1) \, d\mathbf{x}_1 = \lambda_i \psi_i(\mathbf{x}_2) \tag{61}$$

- In practise, the infinite series of 60 must be truncated, yielding a truncated KL approximation

$$\tilde{\mathcal{H}}(\mathbf{x}, \theta) \cong \bar{\mathcal{H}}(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) \psi_i(\mathbf{x}) \tag{62}$$

Karhunen-Loève (KL) expansion

- Gaussian and lognormal random fields have been considered. The covariance function is represented as:

$$\Gamma_{\alpha_Z} = \sigma^2_{\alpha_Z} e^{(-|y_1 - y_2|/b_y) + (-|z_1 - z_2|/b_z)} \tag{63}$$

  where b_y and b_z are the correlation parameters at y and z directions (that corresponds to direction - 1 and direction - 2 respectively). These quantities control the rate at which the covariance decays.
- In a two dimensional physical space the eigensolutions of the covariance function are obtained by solving the integral equation analytically

$$\lambda_i \psi_i(y_2, z_2) = \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \Gamma(y_1, z_1; y_2, z_2) \psi_i(y_1, z_1) dy_1 dz_1 \tag{64}$$

  where $-a_1 \leqslant y \leqslant a_1$ and $-a_2 \leqslant z \leqslant a_2$.
- Assume the eigen-solutions are separable in y and z directions, i.e.

$$\psi_i(y_2, z_2) = \psi_i^{(y)}(y_2) \psi_i^{(z)}(z_2) \tag{65}$$

$$\lambda_i(y_2, z_2) = \lambda_i^{(y)}(y_2) \lambda_i^{(z)}(z_2) \tag{66}$$

Karhunen-Loève (KL) expansion

- The solution of the integral equation reduces to the product of the solutions of two equations of the form

$$\lambda_i^{(y)}\psi_i^{(y)}(y_1) = \int_{-a_1}^{a_1} e^{(-|y_1-y_2|/b_y)}\psi_i^{(y)}(y_2)dy_2 \qquad (67)$$

- The solution of this equation, which is the eigensolution (eigenvalues and eigenfunctions) of an exponential covariance kernel for a one-dimensional random field is obtained as

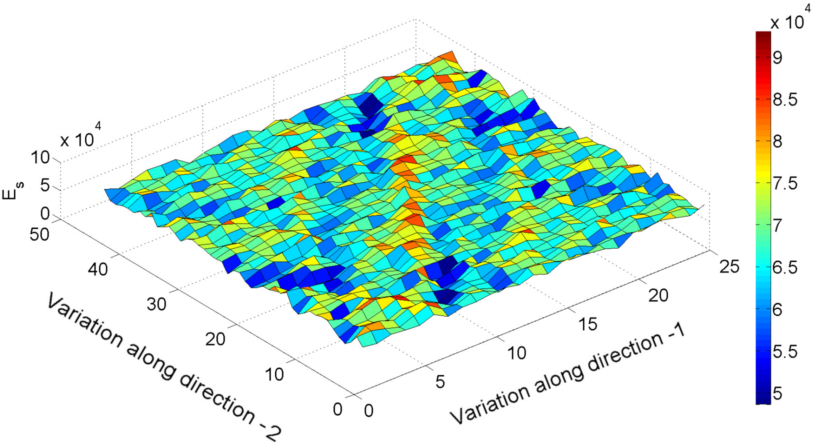
$$\begin{cases} \psi_i(\zeta) = \dfrac{\cos(\omega_i\zeta)}{\sqrt{a+\frac{\sin(2\omega_i a)}{2\omega_i}}} & \lambda_i = \dfrac{2\sigma_{\alpha_z}^2 b}{\omega_i^2+b^2} \quad \text{for } i \text{ odd} \\ \psi_i(\zeta) = \dfrac{\sin(\omega_i^*\zeta)}{\sqrt{a-\frac{\sin(2\omega_i^* a)}{2\omega_i*}}} & \lambda_i^* = \dfrac{2\sigma_{\alpha_z}^2 b}{\omega_i^{*2}+b^2} \quad \text{for } i \text{ even} \end{cases} \qquad (68)$$

where $b = 1/b_y$ or $1/b_z$ and $a = a_1$ or a_2. ζ can be either y or z and ω_i presents the period of the random field.

- The final eigenfunctions are given by

$$\psi_k(y,z) = \psi_i^{(y)}(y)\psi_i^{(z)}(z) \qquad (69)$$

Samples of the random fields



Spatial variability of the intrinsic elastic modulus (E_s) with $\Delta_m = 0.002$

The degree of geometric irregularity

- To define the degree of irregularity, it is assumed that each connecting node of the lattice moves randomly within a certain radius (r_d) around the respective node corresponding to the regular deterministic configuration. For physically realistic variabilities, it is considered that a given node do not cross a neighbouring node, that is

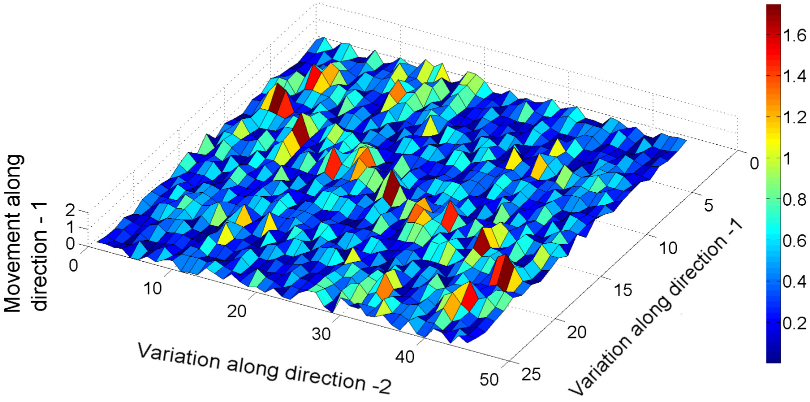
$$r_d < \min\left(\frac{h}{2}, \frac{l}{2}, l\cos\theta\right) \tag{70}$$

- In each realization of the Monte Carlo simulation, all the nodes of the lattice move simultaneously to new random locations within the specified circular bounds. Thus, the degree of irregularity (r) is defined as a non-dimensional ratio of the area of the circle and the area of one regular hexagonal unit as

$$r = \frac{\pi r_d^2 \times 100}{2l\cos\theta(h + l\sin\theta)} \tag{71}$$

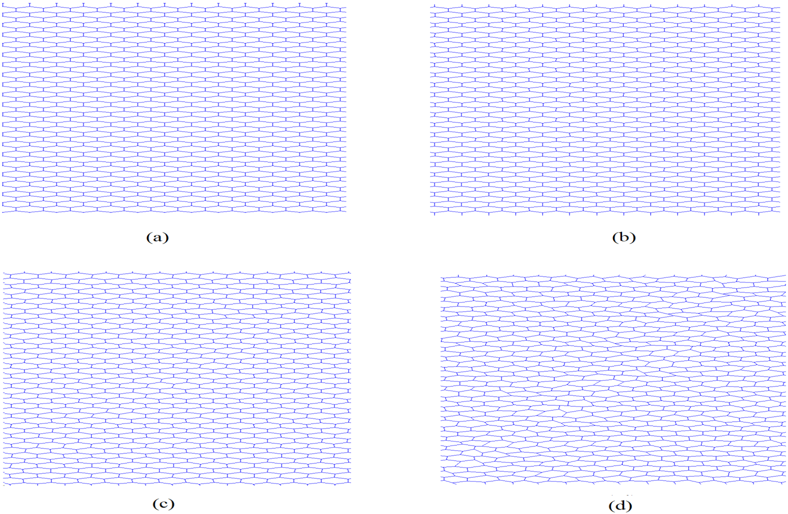
- The degree of irregularity (r) has been expressed as percentage values for presenting the results.

Samples of the random fields



Movement of the top vertices of a tessellating hexagonal unit cell with respect to the corresponding deterministic locations ($r = 6$)

Random geometric configurations



Structural configurations for a single random realisation of an irregular hexagonal lattice considering deterministic cell angle $\theta = 30^{\circ}$ and $h/l = 1$: (a) $r = 0$ (b) $r = 2$ (c) $r = 4$ (d) $r = 6$

Samples of random geometric configurations

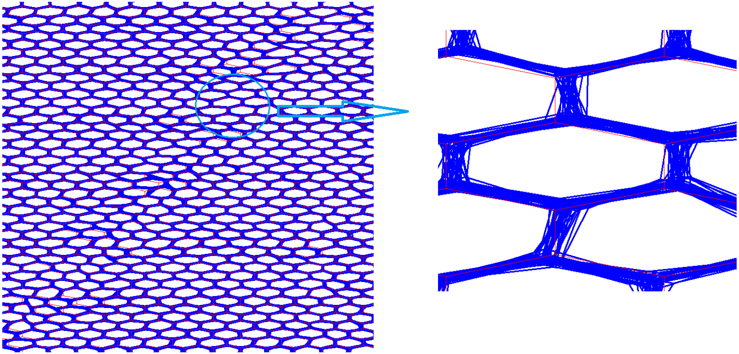
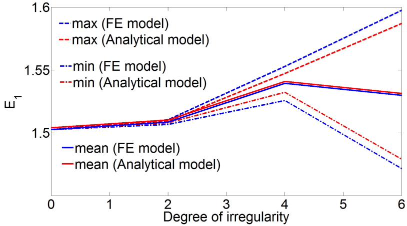


Figure: Simulation bound of the structural configuration of an irregular hexagonal lattice for multiple random realisations considering $\theta = 30^{\circ}$, $h/l = 1$ and $r = 6$. The regular configuration is presented using red colour.

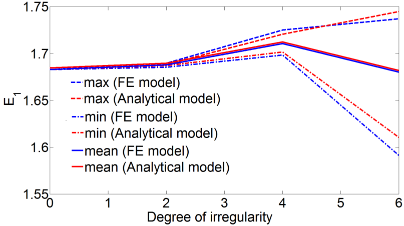
Samples of random geometric configurations

- In randomly inhomogeneous correlated system, spatial variability of the stochastic structural attributes are accounted, wherein each sample of the Monte Carlo simulation includes the spatially random distribution of structural and materials attributes with a rule of correlation.
- The spatial variability in structural and material properties (E_s, μ and ϵ) are physically attributed by degree of structural irregularity (r) and degree of material property variation (Δ_m) respectively.
- As the two Young's moduli and shear modulus for low density lattices are proportional to $E_s\rho^3$ (Zhu et al., 2001), the non-dimensional results for in-plane elastic moduli E_1, E_2, and G_{12}, unless otherwise mentioned, are presented as:
  $\bar{E_1} = \frac{E_{1eq}}{E_s\rho^3}$, $\bar{E_2} = \frac{E_{2eq}}{E_s\rho^3}$
  $\bar{G_{12}} = \frac{G_{12eq}}{E_s\rho^3}$
- ρ is the relative density of the lattice (defined as a ratio of the planar area of solid to the total planar area of the lattice).

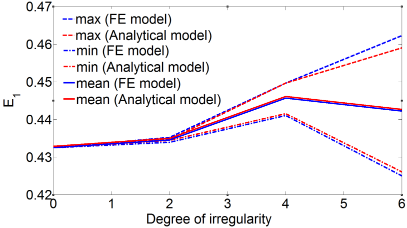
Spatially correlated irregular elastic lattices: E_1



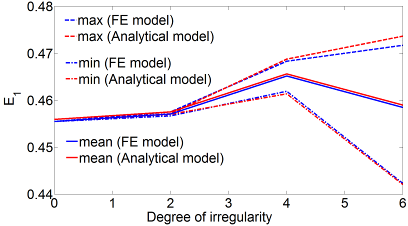
(a) $\theta = 30°; \frac{h}{l} = 1$



(b) $\theta = 30°; \frac{h}{l} = 1.5$

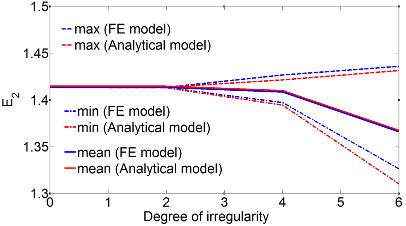


(c) $\theta = 45°; \frac{h}{l} = 1$

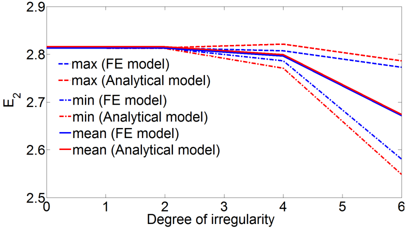


(d) $\theta = 45°; \frac{h}{l} = 1.5$

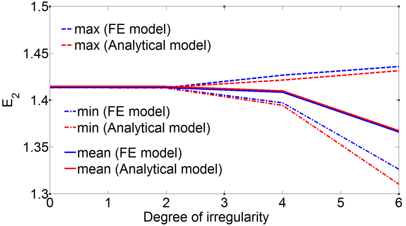
Spatially correlated irregular elastic lattices: E_2



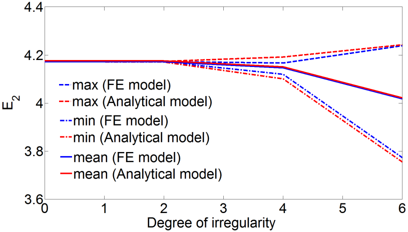
(a) $\theta = 30°; \frac{h}{l} = 1$



(b) $\theta = 30°; \frac{h}{l} = 1.5$

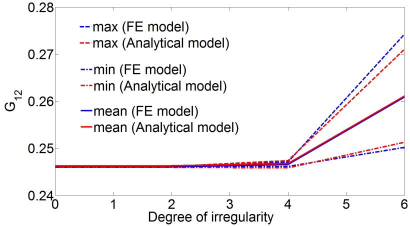


(c) $\theta = 45°; \frac{h}{l} = 1$

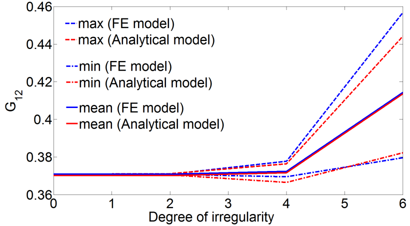


(d) $\theta = 45°; \frac{h}{l} = 1.5$

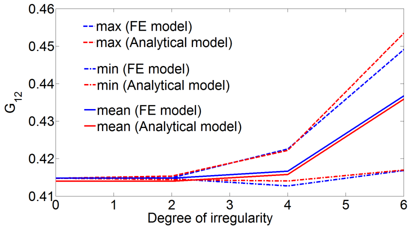
Spatially correlated irregular elastic lattices: G_{12}



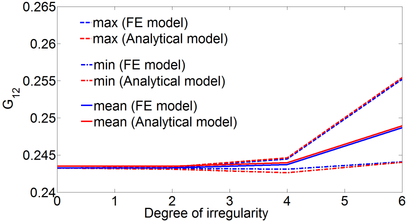
(a) $\theta = 30°; \frac{h}{l} = 1$



(b) $\theta = 30°; \frac{h}{l} = 1.5$

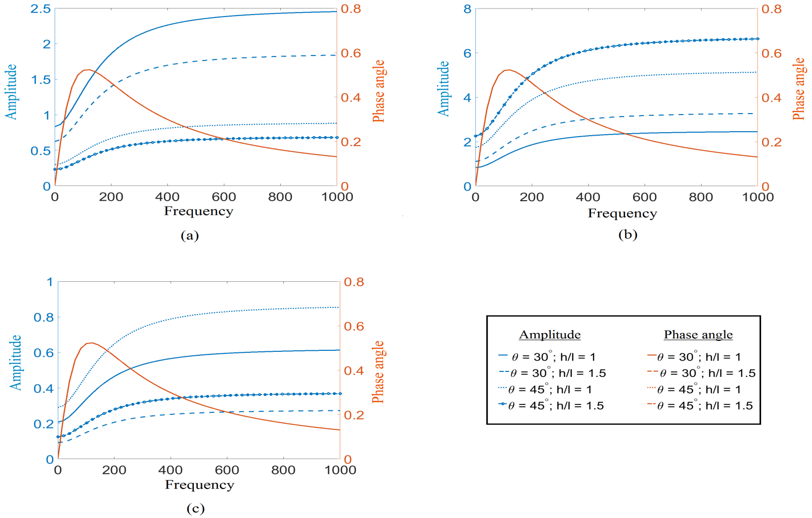


(c) $\theta = 45°; \frac{h}{l} = 1$



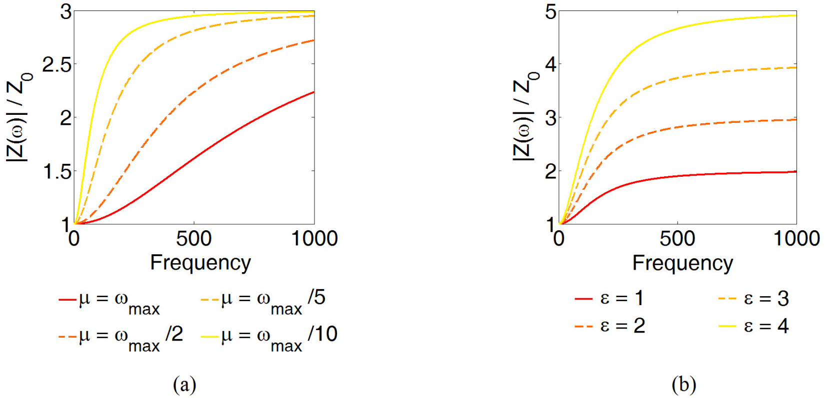
(d) $\theta = 45°; \frac{h}{l} = 1.5$

Viscoelastic properties of regular lattices: E_1, E_2, G_{12}



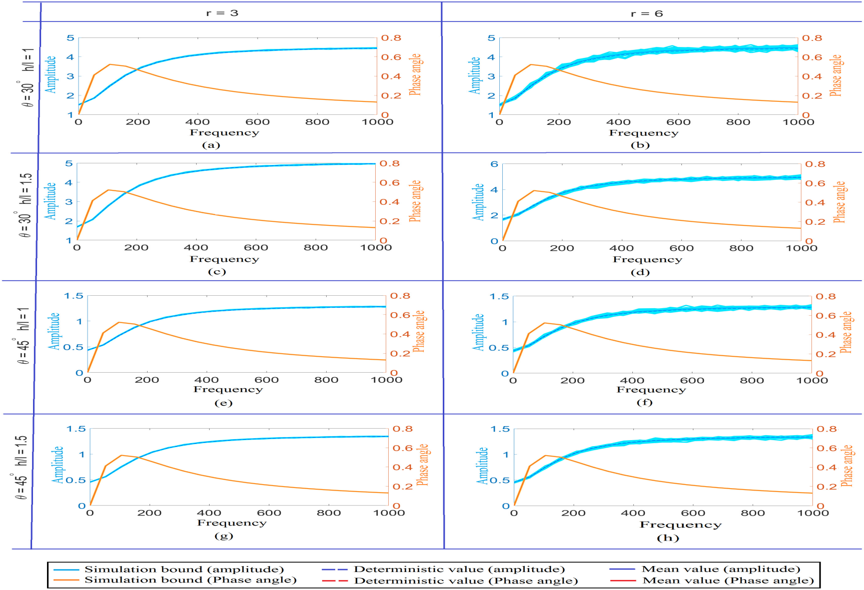
(a) Effect of viscoelasticity on the magnitude and phase angle of E_1 for regular hexagonal lattices (b) Effect of viscoelasticity on the magnitude and phase angle of E_2 for regular hexagonal lattices (c) Effect of viscoelasticity on the magnitude and phase angle of G_{12} for regular hexagonal lattices

Viscoelastic properties of regular lattices

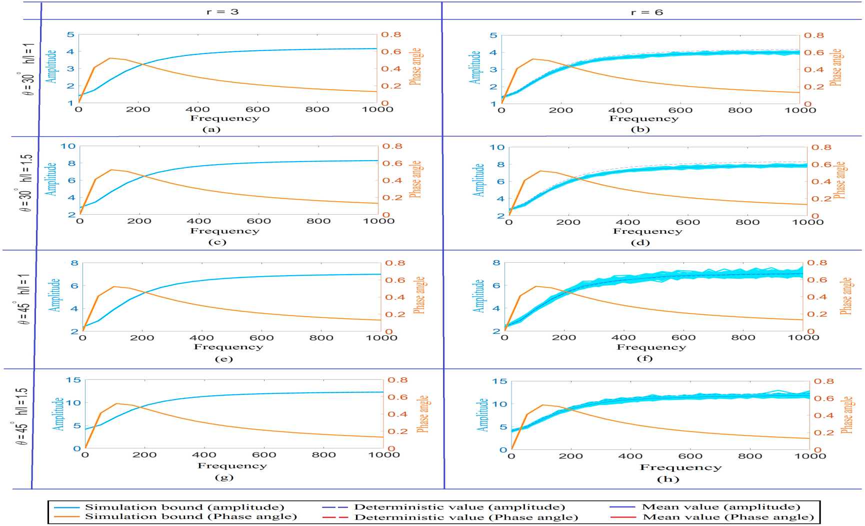


(a) Effect of variation of μ on the viscoelastic modulus of regular hexagonal lattices (considering a constant value of $\epsilon = 2$) (b) Effect of variation of ϵ on the viscoelastic modulus of regular hexagonal lattices (considering a constant value of $\mu = \omega_{max}/5$). Here Z represents the viscoelastic moduli (i.e. E_1, E_2 and G_{12}) and Z_0 is the corresponding elastic modulus value for $\omega = 0$.

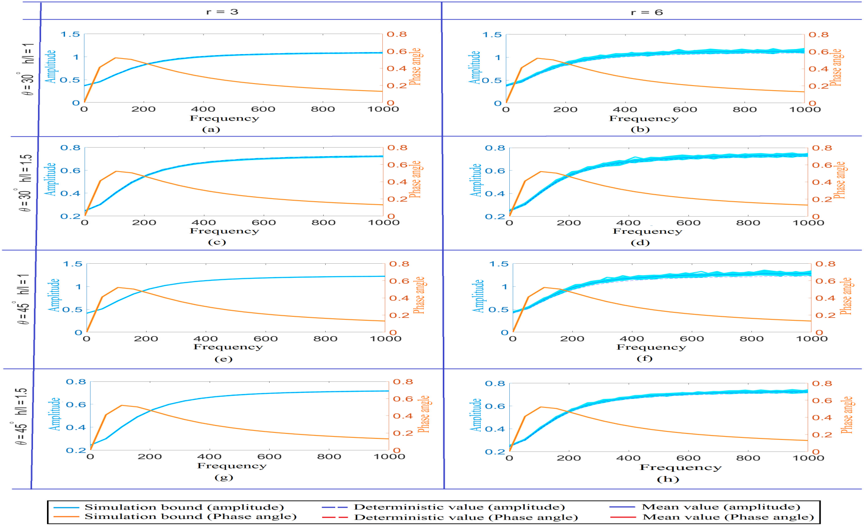
Spatially correlated irregular viscoelastic lattices: E_1



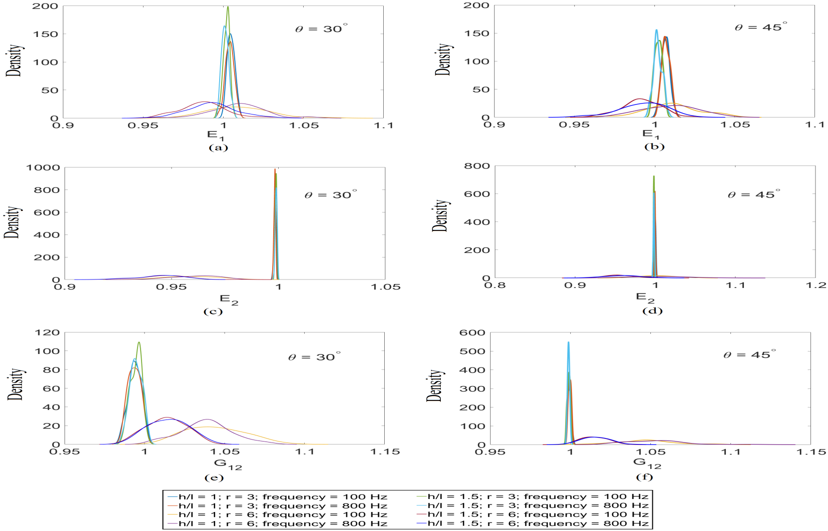
Spatially correlated irregular viscoelastic lattices: E_2



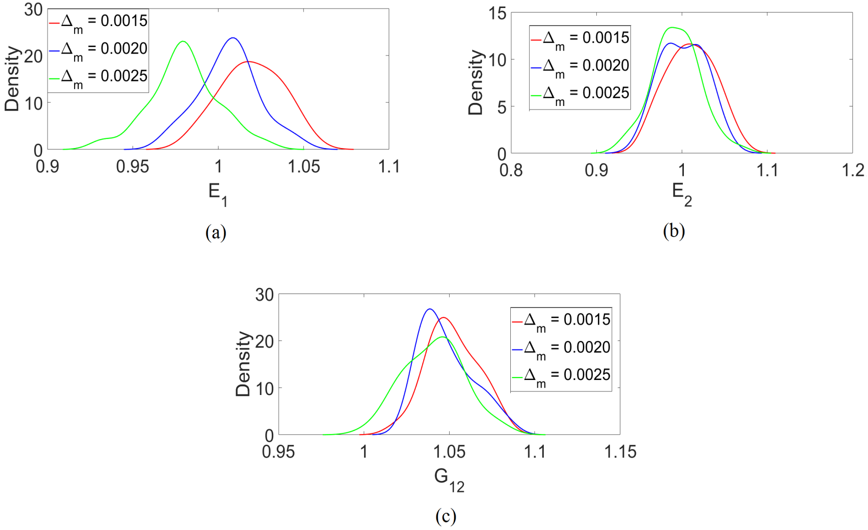
Spatially correlated irregular elastic lattices: G_{12}



Probability density function: random geometry

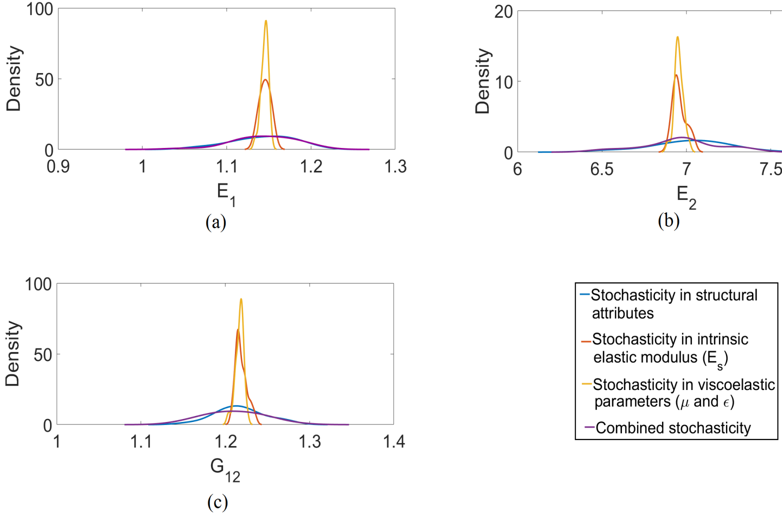


Probability density function: random material property



Probability density function plots for the amplitude of the elastic moduli considering randomly inhomogeneous form of stochasticity for different values of Δ_m (i.e. coefficient of variation for spatially random correlated material properties, such as E_s, μ and ϵ). Results are presented as a ratio of the values corresponding to irregular configurations and respective deterministic values (for a frequency of 800 Hz).

Combined material and geometric uncertianty



Probabilistic descriptions for the amplitudes of three effective viscoelastic properties corresponding to a frequency of 800 Hz considering individual and compound effect of stochasticity in material and structural attributes with $\Delta_{cov} = 0.006$

Conclusions

- The effect of viscoelasticity on irregular hexagonal lattices is investigated in frequency domain considering two different forms of irregularity in structural and material parameters (spatially uncorrelated and correlated).
- Spatially correlated structural and material attributes are considered to account for the effect of randomly inhomogeneous form of irregularity based on Karhunen-Loève expansion.
- The two Young's moduli and shear modulus are dependent on the viscoelastic parameters. Two in-plane Poisson's ratios depend only on structural geometry of the lattice structure.
- The classical closed-form expressions for equivalent in-plane and out of plane elastic properties of regular hexagonal lattice structures have been generalised to consider geometric and material irregularity and viscoelasticity.
- Using the principle of basic structural mechanics on a newly defined unit cell with a homogenisation technique, closed-form expressions have been obtained for E_1, E_2, ν_{12}, ν_{21} and G_{12}.
- The new results reduce to classical formulae of Gibson and Ashby for the special case of no irregularities and no viscoelastic effect.

Closed-form expressions: Elastic Moduli

$$E_{1v}(\omega) = \frac{t^3}{L} \sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)}{\sum_{i=1}^{m} \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij} + l_{2ij} \right) \left(\cos\alpha_{ij} \sin\beta_{ij} - \sin\alpha_{ij} \cos\beta_{ij} \right)^2}{E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(\left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)^2 \right)}} \tag{72}$$

$$E_{2v}(\omega) = \frac{L t^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)}{\sum_{i=1}^{m} E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(l_{3ij}^2 \cos^2\gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) + \frac{l_{1ij}^2 l_{2ij}^2 \left(l_{1ij} + l_{2ij} \right) \cos^2\alpha_{ij} \cos}{\left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)^2} \right.}} \tag{73}$$

$$G_{12v}(\omega) = \frac{L t^3}{\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} \left(l_{1ij} \cos\alpha_{ij} - l_{2ij} \cos\beta_{ij} \right)}{\sum_{i=1}^{m} E_{sij} \left(1 + \epsilon_{ij} \frac{\mathrm{i}\omega}{\mu_{ij} + \mathrm{i}\omega} \right) \left(l_{3ij}^2 \sin^2\gamma_{ij} \left(l_{3ij} + \frac{l_{1ij} l_{2ij}}{l_{1ij} + l_{2ij}} \right) \right)^{-1}}} \tag{74}$$

Closed-form expressions: Poisson's ratios

$$\nu_{12eq} = -\frac{1}{L}\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\cos\alpha_{ij}\cos\beta_{ij}}} \tag{75}$$

$$\nu_{21eq} = -\frac{L}{\sum_{j=1}^{n}\frac{\sum_{i=1}^{m}\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)}{\sum_{i=1}^{m}\frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos\alpha_{ij}\cos\beta_{ij}\left(\cos\alpha_{ij}\sin\beta_{ij} - \sin\alpha_{ij}\cos\beta_{ij}\right)}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{ij}\right)^2\left(l_{3ij}^2\cos^2\gamma_{ij}\left(l_{3ij} + \frac{l_{1ij}l_{2ij}}{l_{1ij}+l_{2ij}}\right) + \frac{l_{1ij}^2 l_{2ij}^2\left(l_{1ij}+l_{2ij}\right)\cos^2\alpha_{ij}}{\left(l_{1ij}\cos\alpha_{ij} - l_{2ij}\cos\beta_{i}\right.}\right.}}} \tag{76}$$

Gibson, L., Ashby, M. F., 1999. Cellular Solids Structure and Properties. Cambridge University Press, Cambridge, UK.
Zhu, H. X., Hobdell, J. R., Miller, W., Windle, A. H., 2001. Effects of cell irregularity on the elastic properties of 2d voronoi honeycombs. Journal of the Mechanics and Physics of Solids 49 (4), 857–870.