Day 1: Introduction to mechanical metamaterials

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SPARC Course: Metamaterial and metasandwich for energy harvesting and vibration control

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Swansea University

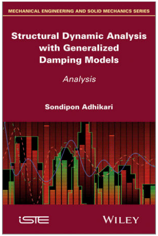
Swansea University

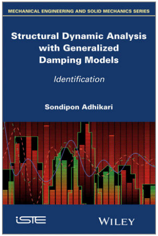
Brief Biography of Professor Adhikari

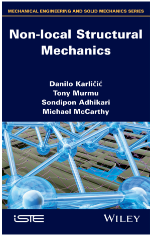
- MSc (Engineering), Indian Institute of Science, Bangalore [08/1995–09/1997]
- PhD in Engineering, University of Cambridge, Trinity College, UK [10/1997–07/2001]
- **Past Positions**: Assistant Prof, Department of Aerospace Engineering, University of Bristol [01/2003–03/2007]
- **Current Positions:** Full Prof (Aerospace Engineering), Swansea University [04/2007–]
- **Awards:** Nehru Cambridge Scholarship; Overseas Research Student Award,; Rouse-Ball Travelling Scholarship from Trinity College, Cambridge; Philip Leverhulme Prize; Associate Fellow of the American Institute of Aeronautics and Astronautics; Wolfson Research Merit Award from the Royal Society, London; EPSRC Ideas factory winner.
- **Professional activities:** Editorial board member of 20 journals, 315 journal papers, $h-$index=66, over 15k citations.

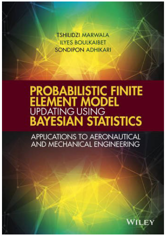
My research interests

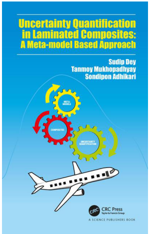
- *Development* of fundamental computational methods for structural dynamics and uncertainty quantification
  A. Dynamics of complex systems
  B. Inverse problems for linear and non-linear dynamics
  C. Uncertainty quantification in computational mechanics
- *Applications* of computational mechanics to emerging multidisciplinary research areas
  D. Vibration energy harvesting / dynamics of wind turbines
  E. Dynamics and mechanics of metamaterials and multi-scale systems











Outline

1. **Overview of the lectures**
2. **Introduction**
3. **A brief history of metamaterials**
4. **Mechanical metamaterials**
   - Homogeneous elastic properties
   - Wave propagation in damped periodic metamaterials
   - Nonlinear metamaterials
   - Experimental methods for mechanical metamaterials
   - Novel designs and demonstration of unusual properties
5. **Cellular metamaterials**
   - Hexagonal lattices
   - Brief Review
6. **Homogeneous elastic properties**

Lectures in this short course

1. *Day 1-A:* Introduction to metamaterials
2. *Day 1-B:* Mechanics of semi-irregular cellular metamaterials
3. Experimental: Hourglass metastructure

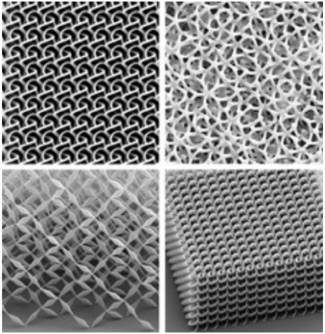
4. *Day 2-A:* Mechanics of irregular cellular viscoelastic metamaterials
5. *Day 2-B:* Dynamic homogenisation of cellular metamaterials
6. *Day 2-C:* Special topics and recent developments
7. Experimental: Nonlinear energy harvesting

Metamaterials

- Mechanical metamaterials have surged to the forefront over the past five years against the backdrop of the unprecedented developments in optical, electromagnetic and acoustic metamaterials.
- Metamaterials are designer media with periodic units comprised of unique tailor-made geometry and pattern aimed at accomplishing exceptional and unusual bulk properties which are unprecedented in conventional materials.
- Therefore, metamaterials are artificial materials designed to outperform naturally occurring materials on various fronts. These include, but are not limited to, electromagnetics, acoustics, optics, infrared, dynamics and mechanical properties. Metamaterials have been established as a highly multidisciplinary research area, which is very active both within and outside the EU.
- `http://www.youtube.com/watch?v=pNsnvRHNfho`
  `https://www.youtube.com/watch?v=-yF4iqbU4qk`

Metamaterials

- Mechanical metamaterials achieve their unusual effective properties from the geometry and structure, and not from the intrinsic property of the constitutive material.
- An essential feature of metamaterials, operating in any frequency ranges or of any length-scales, is the periodicity of a unit cell, as shown in the examples below.



Analysis of metamaterials

- Unlike electromagnetic or acoustics metamaterials, unit cells of mechanical or structural metamaterials tend to be larger as the frequency range of interest is generally lower.
- Consequently, manufacturing methods such as additive layer manufacturing (3D printing) can introduce defects or random disorder, breaking the perfect periodicity within the material.
- This has a profound impact on the analytical and numerical methods characterising mechanical metamaterials as periodicity is a *fundamental requirement* for the majority of the basic techniques (e.g., unit cell methods [1], Floquet-Bloch theorem [2]).
- The consideration of random disorders requires a *probabilistic approach* from the outset. This, in turn, necessitates a differencet in the approaches underpinning analysis and design of mechanical metamaterials.

Analysis of metamaterials

- Another situation when the lack of periodicity of the unit cell within metamaterials can arise is the deliberate introduction of a ‘gradation’ or ‘slow’ variability.
- Such purposeful variabilities can be optimally designed to enhance the certain functionalities of the material.
- In the context of polymer based composites, *functionally graded materials* have shown significant promise and have attracted wide research interests [3]. It is completely conceivable that in the future, functionally graded mechanical metamaterials will be developed.
- New analytical and numerical methods are necessary for mechanical metamaterials.

Historical sketch

- In 1898, **Jagadish Chandra Bose** conducted the first microwave experiment on twisted structures. These twisted structures match the geometries that are known as artificial chiral media in today's terminology. He also researched double refraction (birefringence) in crystals. Other research included polarization of electric field ”waves” that crystals produce. He discovered this type of polarization in other materials including a class of dielectrics.
- The origin of metamaterials was in field of *electromagnetism* and early ideas can be traced back to 1967 [4–6].
- However, it was the seminal paper by Smith et al. [7] in 2000 demonstrating negative permeability and permittivity by a periodic array of split-ring resonators, that started the current interest in metamaterials.
- Since then several concepts and devices have been conceived which challenge conventional physical laws, such as negative refraction [8], the perfect lens [9, 10], and invisibility cloaking in electromagnetism and optics [11, 12].
- These extraordinary developments not only attracted researchers but also captured the imagination of the public and in some cases, science fiction [13].

Historical sketch

- The next round of development was in *acoustic* metamaterials [14, 15] exploiting the idea of locally resonant behaviour [16].
- The consideration of sub-wavelength structures [17] opened up immense possibilities, including negative effective elastic modulus [18], negative density (or mass) [19], or both [20], anisotropy in the effective mass or density [21, 22], and non-reciprocal response [23, 24].
- The *mechanical* metamaterials, the main area of this proposal, emerged following in the footsteps of electromagnetic [25] and acoustic metamaterials [26], primarily within the past five years [27–31].
- Intense research in recent years shows ultralight metamaterials [32] approaching theoretical strength limit [33], pentamode materials [34] with cloaking mode [35], negative refraction elastic waves [36], elastic cloaking [37, 38] and hyperbolic elastic metamaterials [39].

Historical sketch

- The rise of mechanical metamaterials [29, 31] coincides with remarkable recent advances in manufacturing technology [40].
- From the point of view of analytical techniques, two distinct type of metamaterials are considered, namely, (1) static mechanical metamaterials and (2) dynamic mechanical metamaterials.
- The roots of static mechanical metamaterials can be traced back to late 80's with the discovery of negative Poisson's ratio cellular structures [41].
- Dynamic metamaterials differ from static metamaterials by a crucial point - it is the dynamic metamaterials which explicitly exploits sub-wavelength scale properties.
- Both static and dynamic metamaterials will be covered in these lectures.

Homogeneous elastic properties

- Homogenisation methods of periodic elastic materials can be traced back to the classical work by Hashin and Shtrikman [42].
- Such methods for periodic continuum have been developed for piezoelectric inclusions in a recent work by us [43].
- Exploiting periodic boundary conditions and mechanics of a unit cell, closed-form analytical expressions for equivalent elastic moduli for planar 2D cellular materials have been derived in [44–46].
- Homogenisation of metamaterials with sub wavelength dynamics needs to differ from the classical homogenisation approaches due to the fact that there are local resonators embedded in metamaterials.
- This has led to the development of dynamic homogenisation approaches [47–50].
- The dynamic homogenisation can be viewed as a higher-order method [51] compared to the classical static homogenisation approaches.

Homogeneous elastic properties

- Previously mentioned homogenisation approaches are not strictly applicable when the unit cells are not periodic, as will be the case when random inhomogeneities are present in the metamaterial.
- To address this issue, the idea of 'Representative unit cell element (RUCE)' was introduced by the us [52, 53] in the context of static homogenisation of cellular metamaterials (see **??**).
- This approach is a step-change as it provides the geometric basis for considering inhomogeneities in cellular metamaterials and provides closed-form analytical expressions for equivalent (static) elastic properties.
- Homogenisation of continuum systems with random circular inclusions have been discussed recently [54, 55] for static problems.

Wave propagation in damped periodic metamaterials

- Wave propagation in linear periodic structures is a classical topic [56].
- Extensive works have been undertaken since the mid 60's on dynamics of periodic structures [57] in aerospace engineering. The main motivation was to efficiently analyse large aerospace structures made of period units (e.g., periodically stiffened shell in an aircraft fuselage).
- Most of the current computational methods for metamaterials [1] rely on the Floquet-Bloch theorem for wave propagation, which in turn is based on periodic boundary condition for a unit cell.
- Overall wave propagation behaviour depends on the dynamic characteristics of a unit cell and can be understood in terms of the band-gaps [15, 56].
- Consequently, efficient analytical methods [58] and numerical methods for the computation of bandgaps of metamaterials have taken centre stage in most current research [59–61].

Wave propagation in damped periodic metamaterials

- Classical wave propagation approaches were developed for undamped metamaterials.
- Few authors have considered damped metamaterials [62] where internal damping within a unit cell is considered explicitly [63, 64].
- Bandgap analysis for damped metamaterials needs the solution of a complex eigenvalue problem, which poses a computational challenge [65].
- The dynamic behaviour of periodic structures can change drastically due to the presence of disorders [66].
- Experimental works [67] show that in such cases a vibration can localise [68], similar to the phenomenon of Anderson's localisation [69] in atomic crystals.

Nonlinear metamaterials

- The consideration of nonlinear metamaterials [70] requires the analysis of wave propagation in nonlinear periodic structures [71, 72].
- Earlier works in the topic include dynamic analysis of nonlinear lattices [73, 74].
- Nonlinear metamaterials are fundamentally different from their linear counterparts because of the excitation amplitude dependence of the effective properties.
- The research in nonlinear mechanical metamaterials is still in its infancy and experimental or theoretical works directed towards mechanical metamaterials are limited.
- Some authors [75, 76] have proposed perturbation based methods for wave propagation analysis of nonlinear metamaterials.

Nonlinear metamaterials

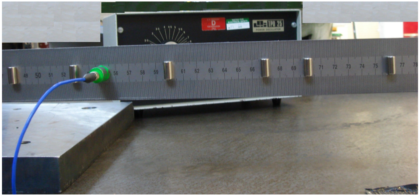
- A mechanical metamaterials concept with local resonators containing negative-stiffness nonlinear springs was analysed in [77].
- An approximated band-gap analysis was reported in [78].
- Experimental works on granular crystals [79, 80] shows several interesting behaviours (e.g., discrete breathers [81], nonlinear normal modes [82]) depending on strength and nature of nonlinearity.
- Reported theoretical works, however, in general, do not explicitly consider damping or random variability
- Future research is necessary in damped nonlinear metamaterials.

Experimental methods for mechanical metamaterials

- Experimental analysis of static properties of cellular metamaterials with regular periodicity has been widely reported [83–85].
- Reported experimental works become relatively scarce when dynamic analysis of mechanical metamaterials are considered.
- There are some experimental works on discrete one-dimensional periodic systems [86, 87].
- Experimental works on continuum periodic systems include dynamics of beams [88] and plates [89].
- An experimental prototype of a metamaterial plate with periodic local resonances was developed and dynamically tested in [90].
- These works do not explicitly consider damping.
- In the context of piezoelectric resonator arrays, damping measurements were experimentally carried out in [91].
- Further work in this direction is however needed, as mentioned in a recent review [1] paper “While experiments have been conducted to probe the effects of damping and nonlinearity in finite structures, the extension to wave motion in periodic media is still at its infancy”. Most of the above experimental works do not consider any disorder.

Experimental methods for mechanical metamaterials

- In one of the earliest experiments on disordered systems, strings with beads placed at random locations [67] demonstrate the existence of mode localisation.
- An experimental study by us [92, 93] on dynamics of a beam with irregularly spaced mass and a plate with random local resonators (see below) show that it is crucial to model and estimate damping for comparing with computational results.




- Experimental work on damped nonlinear metamaterials with random disorders is not yet available. Such gaps in the literature generate a strong need to conduct further studies in this direction.

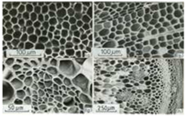
Novel designs and demonstration of unusual properties

- One of the most exciting aspects of mechanical metamaterials is that it challenges conventional beliefs about fundamental properties of materials.
- Theoretical and experimental studies over the past decade have demonstrated incredible properties such as, negative effective elastic modulus [18], negative density [19], ultralight metamaterials [32], pentamode materials [34], elastic cloaking [38], metadamping [63], supratransmission [94] and non-reciprocity [95], to mention a few.
- Physical understandings, engineering design, applications and exploitations of these novel properties are still in their infancy and therefore present immense future opportunity.
- Most of the newly discovered properties are generally based on undamped systems, linear behaviour and perfect periodicity of the unit cells.
- Simultaneous relaxation of these restrictions could see the emergence of further unusual properties yet unseen or not yet conceived

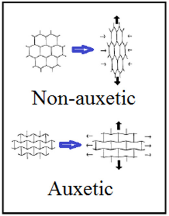
Hexagonal lattices

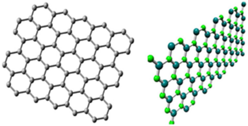
- Hexagonal lattices/ lattice-like structural forms are present as materials and structures in abundance across various length-scales (nano, micro and macro) within natural systems and artificial products.
- For example, honeycombs are widely used as the core of sandwich panels (macro-scale).
- Structure of many woods (like cork and balsa), bone structure, leaf tissues etc. are found to have hexagonal structural forms (micro-scale). There exists a high number of materials with hexagonal structural form at nano-scale (e.g. graphene, hBN etc.[96, 97]).

Hexagonal lattices

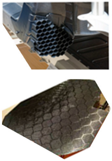


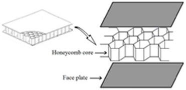
Structure of woods (like cork and balsa), bone structure, leaf tissues, epidermal cells etc; Application in various engineering and medical appliances like smart materials, angioplasty stents, smart bandage etc.





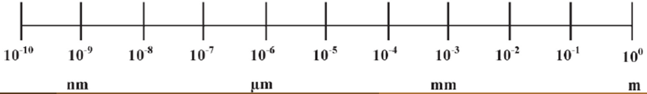
Hexagonal configurations in nano-scale (such as Graphene, Boron Nitride) including molecular auxetics





Sandwich structure with honeycomb core, application in Civil structures

Polymer, metal and ceramic honeycombs for various applications in Mechanical and Aerospace engineering



Hexagonal lattices

The effective mechanical properties of the hexagonal lattices can be controlled/ designed based on their structural configurations.

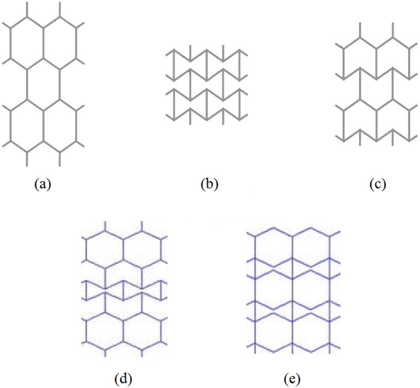


Figure: (a) Hexagonal lattice with positive Poisson's ratio (b) Hexagonal lattice with negative Poisson's ratio (auxetic) (c - e) Hexagonal lattice with zero Poisson's ratio

Hexagonal lattices

- As shown before, the structural configurations for obtaining negative and zero Poisson's ratios are depicted in the figure before.
- The lattice in (a) has conventional positive Poisson's ratio, while by changing the microstructural configuration of the hexagonal lattice intuitively, negative (refer to (b)) and zero Poisson's ratio (refer to (c-e)) can be obtained at a global scale of the lattice.
- In case of a material with negative Poisson's ratio (auxetic), it thickens in the dimensions perpendicular to the direction of stretching and vice-versa, while the dimensions perpendicular to the direction of stretching/ compressing do not change for a material with zero Poisson's ratio.
- Natural materials cannot exhibit such unusual properties that can have various favourable applications in wide range of structural systems.

Hexagonal lattices

- There exists an interesting scope of developing application-specific material microstructures using these lattices.
- Hexagonal structures have been widely investigated as an advanced material because of the capability to meet high performance application-specific demands in various critically desirable parameters such as elastic moduli, specific strength and stiffness, crushing resistance, fatigue strength, acoustic properties, shock absorption properties, electro-mechanical properties, corrosion and fire resistance [44].
- The application of honeycomb cores for lightweight sandwich structures is an active area of research.

Overview of existing works

- To eliminate the need of a detailed finite element modelling for hexagonal lattices/ honeycombs as a part of another complex structural system (host structure such as a sandwich panel), such lattices are generally modelled as a continuous solid medium with equivalent elastic moduli throughout the domain.
- Estimation of effective elastic properties is quite common in the literature of mechanical sciences [98, 99]. A similar approach is followed to evaluate the effective material properties of different nano-structures having hexagonal configurations [100].
- It is a common practice to consider a representative unit cell to model various such periodic structures [101]. Extensive research has been conducted so far to predict effective elastic properties of regular hexagonal lattices without any form of irregularity [44, 45, 102–104].

Overview of existing works

- The unit cell deformation [44]:

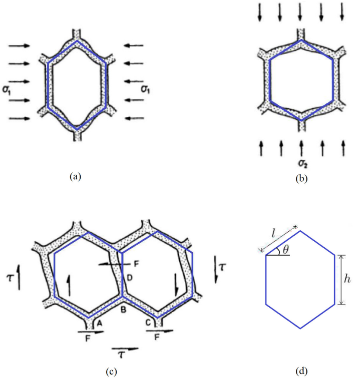


Figure: (a–c) Regular hexagonal honeycombs under the application of three different types of stress (d) Dimensions of a regular honeycomb

- Different modes for the deformation of honeycombs are explained by [105]. One of the most widely accepted analytical formulae for elastic moduli of regular honeycombs are provided by [44].

Homogeneous elastic properties

- The expressions for effective in-plane elastic moduli provided by [44] are listed below

$$E_{1GA} = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{1}$$

$$E_{2GA} = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{2}$$

$$G_{12GA} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{3}$$

$$\nu_{12GA} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta} \tag{4}$$

$$\nu_{21GA} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{5}$$

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