Part 2: Mechanics of semi-irregular cellular metamaterials

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ZCCE

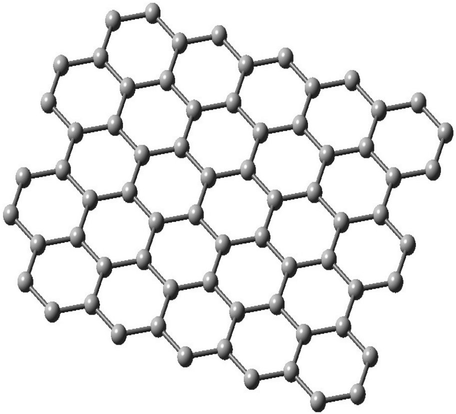
Outline

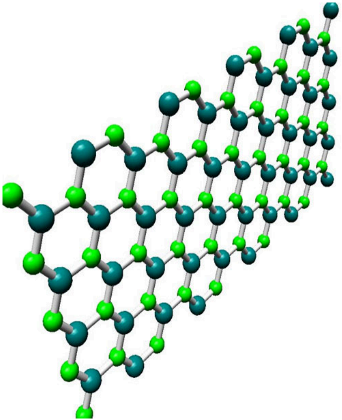
1. **Introduction**
   - Regular Honeycomb
2. **Equivalent elastic properties of random irregular honeycombs**
   - Longitudinal Young's modulus (E_1)
   - Transverse Young's modulus (E_2)
   - Poisson's ratio ν_{12}
   - Poisson's ratio ν_{21}
   - Shear modulus (G_{12})
3. **Uncertainty modelling and simulation**
4. **Numerical results and validation**
   - Numerical results for the homogenised in-plane properties
   - Main observations
5. **Dynamics of sandwich panels with irregular lattice core**
   - Sandwich panel
   - Bending vibration of sandwich panels
   - Derivation of the out of plane shear modulus G_{13}
6. **Conclusions**

Lattice based metamaterials

- Lattice based metamaterials are abundant in man-made and natural systems at various length scales
- Lattice based metamaterials are made of periodic identical/near-identical geometric units
- Among various lattice geometries (triangle, square, rectangle, pentagon, hexagon), hexagonal lattice is most common (note that hexagon is the highest “space filling” pattern in 2D).
- This talk is about in-plane elastic properties of 2D hexagonal lattice structures - commonly known as “honeycombs”

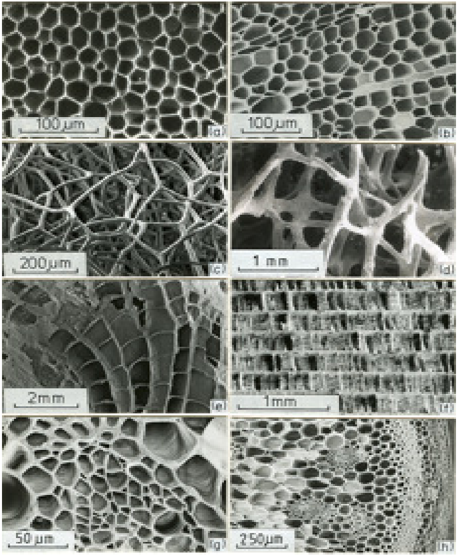
Lattice structures - nano scale

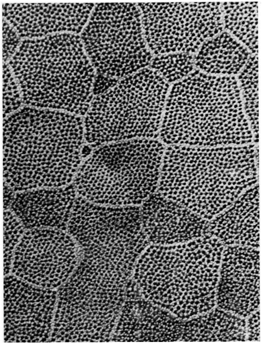




Single layer graphene sheet and born nitride nano sheet

Lattice structures - nature

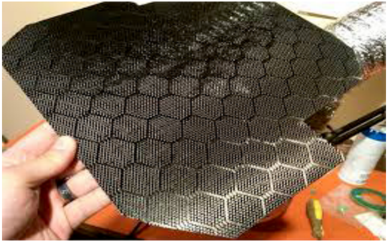




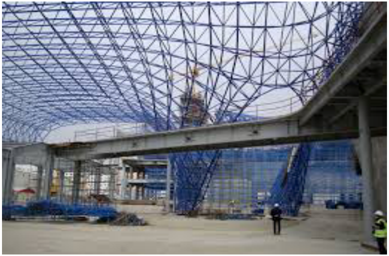
Top left: cork, top right: balsa, next down left: sponge, next down right: trabecular bone, next down left: coral, next down right: cuttlefish bone, bottom left: leaf tissue, bottom right: plant stem, third column - epidermal cells (from web.mit.edu)

Lattice structures - man made

(a) Automotive: BMW i3



(b) Aerospace carbon fibre



(c) Civil engineering: building frame

(d) Architecture

Some questions of general interest

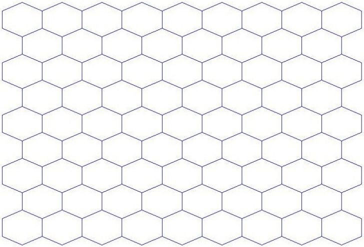
- Shall we consider lattices as “structures” or “materials” from a mechanics point of view?
- At what relative length-scale a lattice *structure* can be considered as a *material* with equivalent elastic properties?
- In what ways structural irregularities “mess up” equivalent elastic properties? Can we evaluate it in a quantitative as well as in a qualitative manner?
- What is the consequence of *random* structural irregularities on the homogenisation approach in general? Can we obtain statistical measures?
- Is there any underlying *ergodic* behaviour for “large” random lattices so that ensemble statistics is close to a sample statistics? How large is “large”?
- How can we efficiently *compute* equivalent elastic properties of random lattice structures?

Regular lattice structures

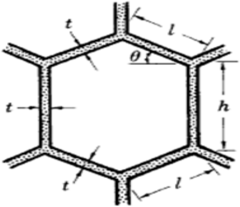
- Honeycombs have been modelled as a continuous solid with an equivalent elastic moduli throughout its domain.
- This approach eliminates the need of detail finite element modelling of honeycombs in complex structural systems like sandwich structures.
- Extensive amount of research has been carried out to predict the equivalent elastic properties of regular honeycombs consisting of perfectly periodic hexagonal cells (El-Sayed et al., 1979; Gibson and Ashby, 1999; Goswami, 2006; Masters and Evans, 1996; Zhang and Ashby, 1992).
- Analysis of two dimensional honeycombs dealing with in-plane elastic properties are commonly based on unit cell approach, which is applicable only for perfectly periodic cellular structures.

Equivalent elastic properties of regular honeycombs

- Unit cell approach - Gibson and Ashby (1999)



(e) Regular hexagon ($\theta = 30^{\circ}$)



(f) Unit cell

- We are interested in homogenised equivalent in-plane elastic properties
- This way, we can avoid a detailed structural analysis considering all the beams and treat it as a material

Equivalent elastic properties of regular honeycombs

- The cell walls are treated as beams of thickness t, depth b and Young's modulus E_s. l and h are the lengths of inclined cell walls having inclination angle θ and the vertical cell walls respectively.
- The equivalent elastic properties are:

$$E_1 = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l} + \sin\theta)\sin^2\theta} \tag{1}$$

$$E_2 = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l} + \sin\theta)}{\cos^3\theta} \tag{2}$$

$$\nu_{12} = \frac{\cos^2\theta}{(\frac{h}{l} + \sin\theta)\sin\theta} \tag{3}$$

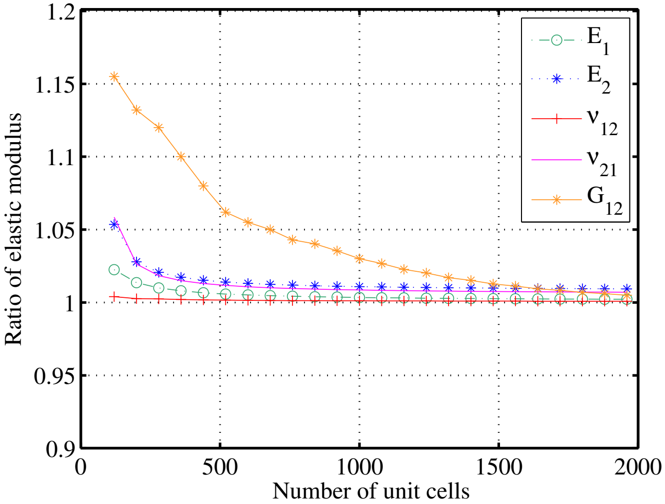
$$\nu_{21} = \frac{(\frac{h}{l} + \sin\theta)\sin\theta}{\cos^2\theta} \tag{4}$$

$$G_{12} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l} + \sin\theta\right)}{\left(\frac{h}{l}\right)^2 (1 + 2\frac{h}{l})\cos\theta} \tag{5}$$

Finite element modelling and verification

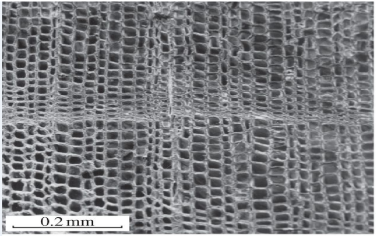
- A finite element code has been developed to obtain the in-plane elastic moduli numerically for honeycombs.
- Each cell wall has been modelled as an Euler-Bernoulli beam element having three degrees of freedom at each node.
- For E_1 and ν_{12}: two opposite edges parallel to direction-2 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-1 is applied while the opposite edge is restrained against translation in direction-1. Remaining two edges (parallel to direction-1) are kept free.
- For E_2 and ν_{21}: two opposite edges parallel to direction-1 of the entire honeycomb structure are considered. Along one of these two edges, uniform stress parallel to direction-2 is applied while the opposite edge is restrained against translation in direction-2. Remaining two edges (parallel to direction-2) are kept free.
- For G_{12}: uniform shear stress is applied along one edge keeping the opposite edge restrained against translation in direction-1 and 2, while the remaining two edges are kept free.

Finite element modelling and verification



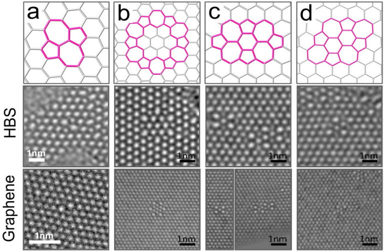
$\theta = 30^\circ$, $h/l = 1.5$. FE results converge to analytical predictions after 1681 cells.

Irregular lattice structures

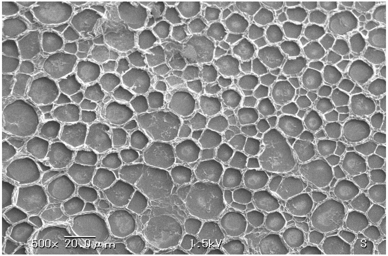


(g) Cedar wood

(h) Manufactured honeycomb core



(i) Graphene image



(j) Fabricated CNT surface

Irregular lattice structures

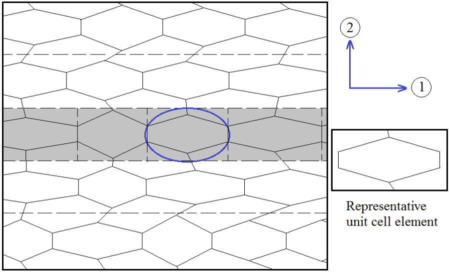
- A significant limitation of the aforementioned unit cell approach is that it cannot account for the spatial irregularity, which is practically inevitable.
- Spatial irregularity in honeycomb may occur due to manufacturing uncertainty, structural defects, variation in temperature, pre-stressing and micro-structural variability in honeycombs.
- To include the effect of irregularity, voronoi honeycombs have been considered in several studies (Li et al., 2005; Zhu et al., 2001, 2006).
- The effect of different forms of irregularity on elastic properties and structural responses of honeycombs are generally based on direct finite element (FE) simulation.
- In the FE approach, a small change in geometry of a single cell may require completely new geometry and meshing of the entire structure. In general this makes the entire process time-consuming and tedious.
- The problem becomes even worse for uncertainty quantification of the responses associated with irregular honeycombs, where the expensive finite element model is needed to be simulated for a large number of samples while using a Monte Carlo based approach.

Irregular lattice structures

- Direct numerical simulation to deal with irregularity in honeycombs may not necessarily provide proper understanding of the underlying physics of the system. An analytical approach could be a simple, insightful, yet an efficient way to obtain effective elastic properties of honeycombs.
- This work develops a structural mechanics based analytical framework for predicting equivalent in-plane elastic properties of irregular honeycomb having spatially random variations in cell angles.
- Closed-form analytical expressions will be derived for equivalent in-plane elastic properties.

The philosophy of the analytical approach for irregular honeycombs

- The key idea to obtain the effective in-plane elastic moduli of the entire irregular honeycomb structure is that it is considered to be consisted of several Representative Unit Cell Elements (RUCE) having different individual elastic moduli.



- The expressions for elastic moduli of a RUCE is derived first and subsequently the expressions for effective in-plane elastic moduli of the entire irregular honeycomb are derived by assembling the individual elastic moduli of these RUCEs using basic principles of mechanics (divide and concur!).

Mathematical idealisation of irregularity in lattice structures

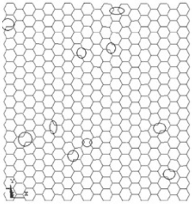


Fig. Randomly missing cell wall

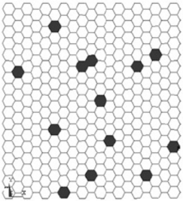


Fig. Random filled cell

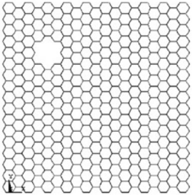
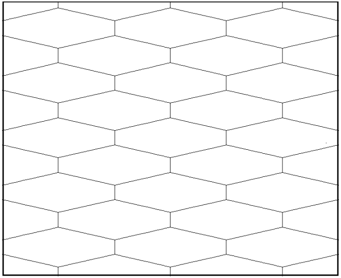
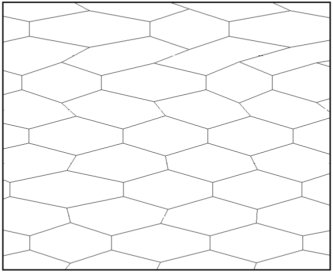


Fig. Missing cell cluster

Irregular honeycomb

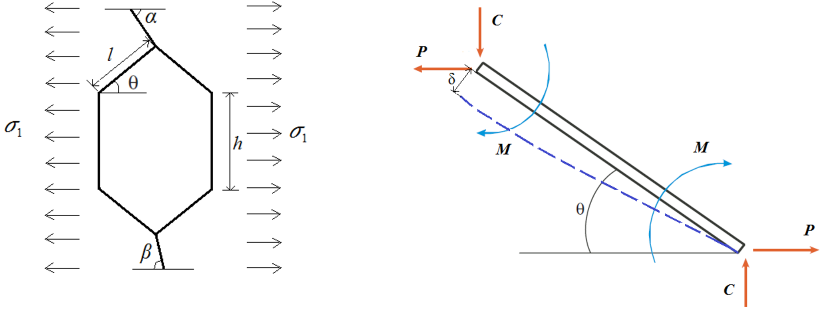




- Random spatial irregularity in cell angle is considered in this study.

Longitudinal Young's modulus (E_1)

- To derive the expression of longitudinal Young's modulus for a RUCE (E_{1U}), stress σ_1 is applied in direction-1 as shown below:



(k) (l)

Figure: RUCE and free-body diagram used in the analysis for E_1

Elastic property of a representative unit cell element (RUCE)

- The inclined cell walls having inclination angle α and β do not have any contribution in the analysis, as the stresses applied on them in two opposite directions neutralise each other. The remaining structure except these two inclined cell walls is symmetric.
- The applied stresses cause the inclined cell walls having inclination angle θ to bend.
- From the condition of equilibrium, the vertical forces C in the free-body diagram of these cell walls need to be zero. The cell walls are treated as beams of thickness t, depth b and Young's modulus E_s. l and h are the lengths of inclined cell walls having inclination angle θ and the vertical cell walls respectively.
- Therefore, we have

$$M = \frac{Pl\sin\theta}{2} \tag{6}$$

where

$$P = \sigma_1(h + l\sin\theta)b \tag{7}$$

Elastic property of a representative unit cell element (RUCE)

- From the standard beam theory, the deflection of one end compared to the other end of the cell wall can be expressed as

$$\delta = \frac{Pl^3 \sin\theta}{12 E_s I} \tag{8}$$

where I is the second moment of inertia of the cell wall, that is $I = bt^3/12$.

- The component of δ parallel to direction-1 is $\delta \sin\theta$. The strain parallel to direction-1 becomes

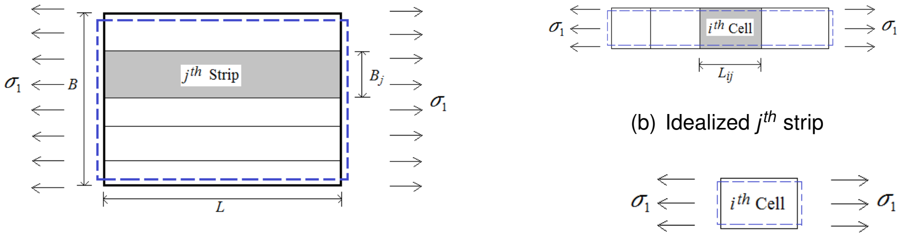
$$\epsilon_1 = \frac{\delta \sin\theta}{l \cos\theta} \tag{9}$$

Thus the Young's modulus in direction-1 for a RUCE can be expressed as

$$E_{1U} = \frac{\sigma_1}{\epsilon_1} = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{\left(\frac{h}{l} + \sin\theta\right)\sin^2\theta} \tag{10}$$

- Next we use this to obtain E_1 for the entire structure.

Elastic property of the entire irregular honeycomb



(a) Entire idealized irregular honeycomb structure

(c) Idealized i^{th} cell in j^{th} strip

Figure: Free-body diagrams of idealized irregular honeycomb structure in the proposed analysis of E_1

- The entire irregular honeycomb structure is assumed to have m and n number of RUCEs in direction-1 and direction-2 respectively. A particular cell having position at i^{th} column and j^{th} row is represented as (i,j), where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

Elastic property of the entire irregular honeycomb

- To obtain E_{1eq}, stress σ_1 is applied in direction-1. If the deformation compatibility condition of j^{th} strip (as highlighted in the figure) is considered, the total deformation due to stress σ_1 of that particular strip (Δ_1) is the summation of individual deformations of each RUCEs in direction-1, while deformation of each of these RUCEs in direction-2 is the same.
- Thus for the j^{th} strip

$$\Delta_1 = \sum_{i=1}^{m} \Delta_{1ij} \tag{11}$$

The equation (11) can be rewritten as

$$\epsilon_1 L = \sum_{i=1}^{m} \epsilon_{1ij} L_{ij} \tag{12}$$

where ϵ_1 and L represent strain and dimension in direction-1 of respective elements.

Elastic property of the entire irregular honeycomb

- Equation (12) leads to

$$\frac{\sigma_1 L}{\hat{E}_{1j}} = \sum_{i=1}^{m} \frac{\sigma_1 L_{ij}}{E_{1Uij}} \tag{13}$$

- From equation (13), equivalent Young's modulus of j^{th} strip ($\hat{E}_{1j}$) can be expressed as

$$\hat{E}_{1j} = \frac{\sum_{i=1}^{m} l_{ij}\cos\theta_{ij}}{\sum_{i=1}^{m} \frac{l_{ij}\cos\theta_{ij}}{E_{1Uij}}} \tag{14}$$

where θ_{ij} is the inclination angle of the cell walls having length l_{ij} in the RUCE positioned at (i,j).

Elastic property of the entire irregular honeycomb

- After obtaining the Young's moduli of n number of strips, they are assembled to achieve the equivalent Young's modulus of the entire irregular honeycomb structure (E_{1eq}) using force equilibrium and deformation compatibility conditions.

$$\sigma_1 Bb = \sum_{j=1}^{n} \sigma_{1j} B_j b \tag{15}$$

where B_j is the dimension of j^{th} strip in direction-2 and $B = \sum_{j=1}^{n} B_j$. b represents the depth of honeycomb.

- As strains in direction-1 for each of the n strips are same to satisfy the deformation compatibility condition, equation (15) leads to

$$\left(\sum_{j=1}^{n} B_j \right) E_{1eq} = \sum_{j=1}^{n} \hat{E}_{1j} B_j \tag{16}$$

Elastic property of the entire irregular honeycomb

- Using equation (14) and equation (16), equivalent Young's modulus in direction-1 of the entire irregular honeycomb structure (E_{1eq}) can be expressed as:

Equivalent E_1

$$E_{1eq} = \frac{1}{\sum_{j=1}^{n} B_j} \sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} \frac{l_{ij} \cos\theta_{ij}}{E_{1Uij}}} \right) B_j \tag{17}$$

- Here Young's modulus in direction-1 of a RUCE positioned at (i,j) is E_{1Uij}, which can be obtained from equation (10) as

$$E_{1Uij} = E_s \left(\frac{t}{l_{ij}} \right)^3 \frac{\cos\theta_{ij}}{\left(\frac{h}{l_{ij}} + \sin\theta_{ij} \right) \sin^2\theta_{ij}} \tag{18}$$

Transverse Young's modulus (E_2)

- To derive the expression of transverse Young's modulus for a RUCE (E_{2U}), stress σ_2 is applied in direction-2 as shown below:

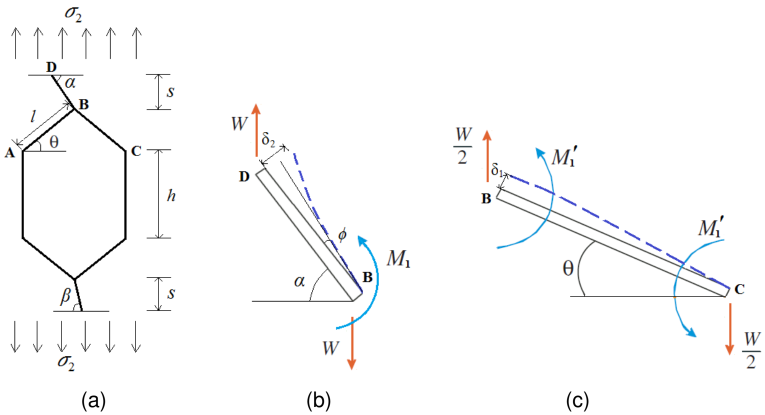


Figure: RUCE and free-body diagram used in the proposed analysis for E_2

Elastic property of a representative unit cell element (RUCE)

- Total deformation of the RUCE in direction-2 consists of three components, namely deformation of the cell wall having inclination angle α, deformation of the cell walls having inclination angle θ and deformation of the cell wall having inclination angle β.
- If the remaining structure except the two inclined cell walls having inclination angle α and β is considered, two forces that act at joint B are W and M_1. For the cell wall having inclination angle α, effect of the bending moment M_1 generated due to application of W at point D is only to create rotation (ϕ) at the joint B.
- Vertical deformation of the cell wall having inclination angle α has two components, bending deformation in direction-2 and rotational deformation due the rotation of joint B.

Elastic property of a representative unit cell element (RUCE)

- After some algebra and mechanics, the total deformation in direction-2 of the RUCE due to the application of stresses σ_2 is

$$\delta_v = \frac{\sigma_2 l \cos\theta}{E_s t^3} \left(2l^3 \cos^2\theta + 8s^3 \left(\frac{\cos^2\alpha}{\sin^3\alpha} + \frac{\cos^2\beta}{\sin^3\beta} \right) + 2s^2 l (\cot^2\alpha + \cot^2\beta) \right) \quad (19)$$

- The strain in direction-2 can be obtained as

$$\epsilon_2 = \frac{\delta_v}{h + 2s + 2l\sin\theta} \quad (20)$$

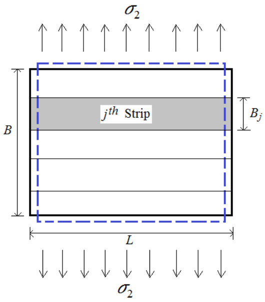
Elastic property of a representative unit cell element (RUCE)

- Therefore, the Young's modulus in direction-2 of a RUCE can be expressed as

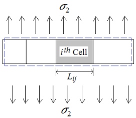
$$E_{2U} = \frac{\sigma_2}{\epsilon_2} = E_s \left(\frac{t}{l}\right)^3 \times \frac{\left(\frac{h}{l} + 2\frac{s}{l} + 2\sin\theta\right)}{\cos\theta\left(2\cos^2\theta + 8\left(\frac{s}{l}\right)^3\left(\frac{\cos^2\alpha}{\sin^3\alpha} + \frac{\cos^2\beta}{\sin^3\beta}\right) + 2\left(\frac{s}{l}\right)^2(\cot^2\alpha + \cot^2\beta)\right)} \quad (21)$$

Elastic property of the entire irregular honeycomb

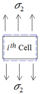
- To derive the expression of equivalent Young's modulus in direction-2 for the entire irregular honeycomb structure (E_{2eq}), the Young's moduli for the constituting RUCEs (E_{2U}) are "assembled".



(a) Entire idealized irregular honeycomb structure



(b) Idealized j^{th} strip



(c) Idealized i^{th} cell in j^{th} strip

Figure: Free-body diagrams of idealized irregular honeycomb structure

Elastic property of the entire irregular honeycomb

- When the force equilibrium under the application of stress σ_2 of j^{th} strip (as highlighted in 4(b)) is considered:

$$\sigma_2 \left(\sum_{i=1}^{m} 2 l_{ij} \cos\theta_{ij} \right) b = \left(\sum_{i=1}^{m} \sigma_{2ij} 2 l_{ij} \cos\theta_{ij} \right) b \qquad (22)$$

- By deformation compatibility condition, strains of each RUCE in direction-2 of the j^{th} strip are same. Equation (22), rewritten as

$$\hat{E_{2j}} \left(\sum_{i=1}^{m} l_{ij} \cos\theta_{ij} \right) \epsilon = \left(\sum_{i=1}^{m} E_{2Uij} l_{ij} \cos\theta_{ij} \epsilon_{ij} \right) \qquad (23)$$

  where $\epsilon_{ij} = \epsilon$, for $i = 1, 2...m$ in the j^{th} strip.

- $\hat{E_{2j}}$ is the equivalent elastic modulus in direction-2 of the j^{th} strip:

$$\hat{E_{2j}} = \frac{\sum_{i=1}^{m} E_{2Uij} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}} \qquad (24)$$

Elastic property of the entire irregular honeycomb

- Total deformation of the entire honeycomb in direction-2 (Δ_2) is the sum of deformations of each strips in that direction,

$$\Delta_2 = \sum_{j=1}^{n} \Delta_{2ij} \tag{25}$$

- Equation (25) can be rewritten as

$$\epsilon_2 B = \sum_{j=1}^{n} \epsilon_{2j} B_j \tag{26}$$

  where ϵ_2, ϵ_{2j} and B_j represent total strain of the entire honeycomb structure in direction-2, strain of j^{th} strip in direction-2 and dimension in direction-2 of j^{th} strip respectively.
- From equation (26) we have

$$\frac{\sigma_2 \sum\limits_{j=1}^{n} B_j}{E_{2eq}} = \sum_{j=1}^{n} \frac{\sigma_2 B_j}{\hat{E}_{2j}} \tag{27}$$

Elastic property of the entire irregular honeycomb

- From equation (24) and equation (27), the Young's modulus in direction-2 of the entire irregular honeycomb structure can be expressed as

Equivalent E_2

$$E_{2eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} E_{2Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j \tag{28}$$

- Here Young's modulus in direction-2 of a RUCE positioned at (i,j) is E_{2Uij}, which can be obtained from equation (21).

Special case: classical deterministic results

- The expressions of Young's moduli for randomly irregular honeycombs (equation (17) and (28)) reduces to the formulae provided by Gibson and Ashby (Gibson and Ashby, 1999) in case of uniform honeycombs (i.e. $B_1 = B_2 = ... = B_n$; $s = h/2$; $\alpha = \beta = 90^\circ$; $l_{ij} = l$ and $\theta_{ij} = \theta$, for all i and j).
- By applying the conditions $B_1 = B_2 = ... = B_n$; $l_{ij} = l$ and $\theta_{ij} = \theta$, equation (17) and (28) reduce to E_{1U} and E_{2U} respectively.
- For $s = h/2$ and $\alpha = \beta = 90^\circ$, E_{1U} and E_{2U} produce the same expressions for Young's moduli of uniform honeycomb as presented by Gibson and Ashby (Gibson and Ashby, 1999).
- In the case of regular uniform honeycombs ($\theta = 30^\circ$)

$$\frac{E_1^*}{E_s} = \frac{E_2^*}{E_s} = 2.3 \left(\frac{t}{l}\right)^3 \tag{29}$$

where E_1^* and E_2^* denote the Young moduli of uniform regular honeycombs in longitudinal and transverse direction respectively.

Poisson's ratio ratio ν_{12}

- Poisson's ratios are calculated by taking the negative ratio of strains normal to, and parallel to, the loading direction.
- Poisson's ratio of a RUCE for the loading direction-1 (ν_{12U}) is obtained as

$$\nu_{12U} = -\frac{\epsilon_2}{\epsilon_1} \tag{30}$$

  where ϵ_1 and ϵ_2 represent the strains of a RUCE in direction-1 and direction-2 respectively due to loading in direction-1.
- ϵ_1 can be obtained from equation (9). From 1(l), ϵ_2 can be expressed as

$$\epsilon_2 = -\frac{2\delta \cos\theta}{h + 2l\sin\theta + 2s} \tag{31}$$

- Thus the expression for Poisson's ratio of a RUCE for the loading direction-1 becomes

$$\nu_{12U} = \frac{2\cos^2\theta}{\left(2\sin\theta + 2\frac{s}{l} + \frac{h}{l}\right)\sin\theta} \tag{32}$$

Poisson's ratio of the entire irregular honeycomb

- To derive the expression of equivalent Poisson's ratio for loading direction-1 of the entire irregular honeycomb structure (ν_{12eq}), the Poisson's ratios for the constituting RUCEs (ν_{12U}) are "assembled".
- For obtaining ν_{12eq}, stress σ_1 is applied in direction-1. If the application of stress σ_1 in the j^{th} strip is considered, total deformation of the j^{th} strip in direction-1 is summation of individual deformations of the RUCEs in direction-1 of that particular strip.
- Thus from equation (12), using the basic definition of ν_{12},

$$-\frac{\epsilon_2}{\hat{\nu}_{12j}}L = -\sum_{i=1}^{m}\frac{\epsilon_{2ij}L_{ij}}{\nu_{U12ij}} \tag{33}$$

  where ϵ_2 and ϵ_{2ij} are the strains in direction-2 of j^{th} strip and individual RUCEs of j^{th} strip respectively.
- ν_{U12ij} represents the Poisson's ratio for loading direction-1 of a RUCE positioned at (i,j). $\hat{\nu}_{12j}$ denotes the equivalent Poisson's ratio for loading direction-1 of the j^{th} strip.

Poisson's ratio of the entire irregular honeycomb

- Ensuring the deformation compatibility condition $\epsilon_2 = \epsilon_{2ij}$ for $i = 1, 2, ..., m$ in the j^{th} strip, equation (33) leads to

$$\hat{\nu}_{12j} = \frac{L}{\displaystyle\sum_{i=1}^{m} \frac{L_{ij}}{\nu_{12Uij}}} \tag{34}$$

- Total deformation of the entire honeycomb structure in direction-2 under the application of stress σ_1 along the two opposite edges parallel to direction-2 is summation of the individual deformations in direction-2 of n number of strips. Thus

$$\epsilon_2 B = \sum_{j=1}^{n} \epsilon_{2j} B_j \tag{35}$$

Poisson's ratio of the entire irregular honeycomb

- Using the basic definition of ν_{12} equation (35) becomes

$$\nu_{12eq}\epsilon_1 B = \sum_{j=1}^{n} \nu_{12j}\epsilon_{1j} B_j \tag{36}$$

  where ν_{12eq} represents the equivalent Poisson's ratio for loading direction-1 of the entire irregular honeycomb structure.
- ϵ_1 and ϵ_{1j} denote the strain of entire honeycomb structure in direction-1 and strain of j^{th} strip in direction-1 respectively.
- From the condition of deformation comparability $\epsilon_1 = \epsilon_{1j}$ for $j = 1, 2, ..., n$. Thus from equation (34) and equation (36):

Equivalent ν_{12}

$$\nu_{12eq} = \frac{1}{\sum\limits_{j=1}^{n} B_j} \sum_{j=1}^{n} \left(\frac{\sum\limits_{i=1}^{m} l_{ij}\cos\theta_{ij}}{\sum\limits_{i=1}^{m} \frac{l_{ij}\cos\theta_{ij}}{\nu_{12Uij}}} \right) B_j \tag{37}$$

Here ν_{12Uij} can be obtained from equation (32).

Poisson's ratio ν_{21}

- Poisson's ratio of a RUCE for the loading direction-2 (ν_{21U}) is obtained as

$$\nu_{21U} = -\frac{\epsilon_1}{\epsilon_2} \tag{38}$$

  where ϵ_1 and ϵ_2 represent the strains of a RUCE in direction-1 and direction-2 respectively due to loading in direction-2.

- ϵ_2 can be obtained from equation (19) and equation (20) as

$$\epsilon_2 = \frac{\sigma_2 l \cos\theta}{E_s t^3(h + 2s + 2l\sin\theta)} \left(2l^3\cos^2\theta + 8s^3\left(\frac{\cos^2\alpha}{\sin^3\alpha} + \frac{\cos^2\beta}{\sin^3\beta}\right) + 2s^2 l(cot^2\alpha + cot^2\beta)\right) \tag{39}$$

- We have

$$\epsilon_1 = -\frac{\delta_1 \sin\theta}{l\cos\theta} \tag{40}$$

with $\delta_1 = \dfrac{\left(\dfrac{W}{2}\cos\theta\right) l^3}{12E_s l}$ and $W = 2\sigma_2 lb\cos\theta$.

Poisson's ratio of a representative unit cell element (RUCE)

- Thus equation (40) reduces to

$$\epsilon_1 = -\frac{\sigma_2 l^3 \sin\theta \cos\theta}{E_s t^3} \tag{41}$$

- Thus the expression for Poisson's ratio of a RUCE for the loading direction-2 becomes

$$\nu_{21U} = \frac{\sin\theta\left(\frac{h}{l} + 2\frac{s}{l} + 2\sin\theta\right)}{2\cos^2\theta + 8\left(\frac{s}{l}\right)^3\left(\frac{\cos^2\alpha}{\sin^3\alpha} + \frac{\cos^2\beta}{\sin^3\beta}\right) + 2\left(\frac{s}{l}\right)^2(\cot^2\alpha + \cot^2\beta)} \tag{42}$$

Poisson's ratio of the entire irregular honeycomb

- To derive the expression of equivalent Poisson's ratio for loading direction-2 of the entire irregular honeycomb structure (ν_{21eq}), the Poisson's ratios for the constituting RUCEs (ν_{21U}) are assembled.
- For obtaining ν_{21eq}, stress σ_2 is applied in direction-2. If the application of stress σ_2 in the j^{th} strip is considered, total deformation of the j^{th} strip in direction-1 is summation of individual deformations of the RUCEs in direction-1 of that particular strip. Thus,

$$\epsilon_1 L = \sum_{i=1}^{m} \epsilon_{1ij} L_{ij} \quad (43)$$

- Using the basic definition of ν_{21} equation (43) leads to

$$\hat{\nu}_{21j} \epsilon_2 L = \sum_{i=1}^{m} \nu_{21Uij} \epsilon_{2ij} L_{ij} \quad (44)$$

where $\hat{\nu}_{21j}$ represents the equivalent Poisson's ratio for loading direction-2 of the j^{th} strip.

Poisson's ratio of the entire irregular honeycomb

- ϵ_2 and ϵ_{2ij} are the strains in direction-2 of j^{th} strip and individual RUCEs of j^{th} strip respectively. ν_{21Uij} represents the Poisson's ratio for loading direction-2 of a RUCE positioned at (i,j).
- To ensure the deformation compatibility condition $\epsilon_2 = \epsilon_{2ij}$ for $i = 1, 2, ..., m$ in the j^{th} strip. Thus equation (44) leads to

$$\hat{\nu_{21j}} = \frac{\sum_{i=1}^{m} \nu_{21Uij} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}} \tag{45}$$

- Total deformation of the entire honeycomb structure in direction-2 under the application of stress σ_2 along the two opposite edges parallel to direction-1 is summation of the individual deformations in direction-2 of n number of strips. Thus

$$\epsilon_2 B = \sum_{j=1}^{n} \epsilon_{2j} B_j \tag{46}$$

Poisson's ratio of the entire irregular honeycomb

- By definition of ν_{21} equation (46) leads to

$$\frac{\epsilon_1}{\nu_{21eq}} B = \sum_{j=1}^{n} \frac{\epsilon_{1j}}{\hat{\nu_{21j}}} B_j \tag{47}$$

- From the condition of deformation comparability $\epsilon_1 = \epsilon_{1j}$ for $j = 1, 2, ..., n$. Thus the equivalent Poisson's ratio for loading direction-2 of the entire irregular honeycomb structure:

Equivalent ν_{12}

$$\nu_{21eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} \nu_{21Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j \tag{48}$$

Here ν_{21Uij} can be obtained from equation (42).

Shear modulus (G_{12})

- To derive the expression of shear modulus (G_{12U}) for a RUCE, shear stress τ is applied as shown below:

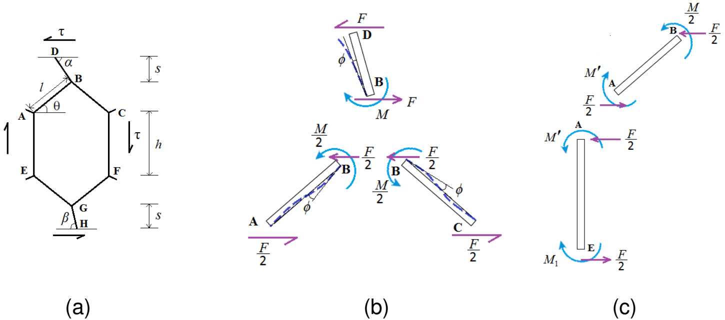


Figure: RUCE and free-body diagram used in the proposed analysis for G_{12}

Elastic property of a representative unit cell element (RUCE)

- Total lateral movement of point D with respect to point H can be obtained as

$$\delta_L = \frac{2\tau l \cos\theta}{Et^3}\left(2ls^2 + \frac{h^3}{2} + 4s^3\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta}\right) + (s + l\sin\theta)h^2\right) \quad (49)$$

- The shear strain γ for a RUCE can be expressed as

$$\gamma = \frac{\delta_L}{2s + h + 2l\sin\theta} = \frac{2\tau l \cos\theta}{Et^3(2s + h + 2l\sin\theta)} \times \left(2ls^2 + \frac{h^3}{2} + 4s^3\left(\frac{1}{\sin\alpha} + \frac{1}{\sin\beta}\right) + (s + l\sin\theta)h^2\right) \quad (50)$$

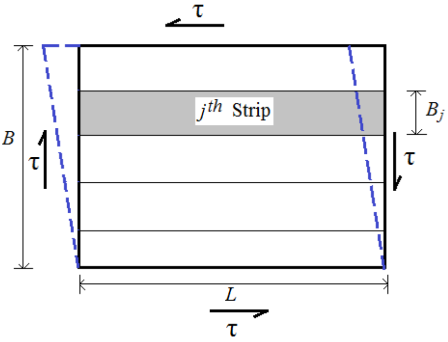
Elastic property of a representative unit cell element (RUCE)

- Thus the expression for shear modulus of a RUCE becomes

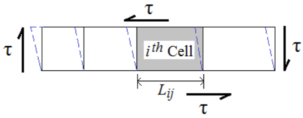
$$G_{12U} = \frac{\tau}{\gamma} = E_s \left(\frac{t}{l}\right)^3 \times \frac{\left(2\frac{s}{l} + \frac{h}{l} + 2\sin\theta\right)}{2\cos\theta\left(2\left(\frac{s}{l}\right)^2 + 4\left(\frac{s}{l}\right)^3\left(\frac{1}{sin\alpha} + \frac{1}{sin\beta}\right) + \frac{1}{2}\left(\frac{h}{l}\right)^3 + \left(\frac{s}{l} + \sin\theta\right)\left(\frac{h}{l}\right)^2\right)} \quad (51)$$

Elastic property of the entire irregular honeycomb

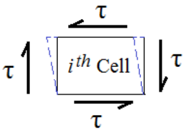
- To derive the expression of equivalent shear modulus of the entire irregular honeycomb structure (G_{12eq}), the shear moduli for the constituting RUCEs (G_{12U}) are "assembled":



(a) Entire idealized irregular honeycomb structure



(b) Idealized j^{th} strip



(c) Idealized i^{th} cell in j^{th} strip

Figure: Free-body diagrams of idealized irregular honeycomb structure in the proposed analysis of G_{12}

Elastic property of the entire irregular honeycomb

- For obtaining G_{12eq}, shear stress τ is applied parallel to direction direction-1. If the equilibrium of forces for application of stress τ in the j^{th} strip is considered:

$$\tau L = \sum_{i=1}^{m} \tau_{ij} L_{ij} \quad (52)$$

- By definition of shear modulus equation (52) can be rewritten as

$$\hat{G}_{12j} \gamma L = \sum_{i=1}^{m} G_{12Uij} \gamma_{ij} L_{ij} \quad (53)$$

  where $\hat{G}_{12j}$ represents the equivalent shear modulus of the j^{th} strip.
- γ and γ_{ij} are the shear strains of j^{th} strip and individual RUCEs of the j^{th} strip respectively. G_{12Uij} represents the shear modulus of a RUCE positioned at (i,j).

Elastic property of the entire irregular honeycomb

- To ensure the deformation compatibility condition $\gamma = \gamma_{ij}$ for $i = 1, 2, ..., m$ in the j^{th} strip. Thus equation (53) leads to

$$\hat{G}_{12j} = \frac{\sum_{i=1}^{m} G_{12Uij} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}} \tag{54}$$

- Total lateral deformation of one edge compared to the opposite edge of the entire honeycomb structure under the application of shear stress τ is the summation of the individual lateral deformations of n number of strips. Thus

$$\gamma B = \sum_{j=1}^{n} \gamma_j B_j \tag{55}$$

- By definition of G_{12} equation (55) leads to

$$\frac{\tau}{G_{12eq}} B = \sum_{j=1}^{n} \frac{\tau_j}{\hat{G_{12j}}} B_j \tag{56}$$

Elastic property of the entire irregular honeycomb

- From equation (54) and (56), the equivalent shear modulus of the entire irregular honeycomb structure can be expressed as:

Equivalent G_{12}

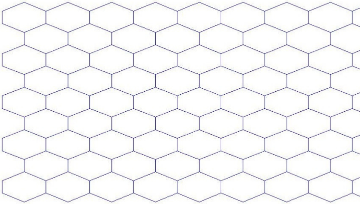
$$G_{12eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} G_{12Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j \tag{57}$$

Here G_{12Uij} can be obtained from equation (51).

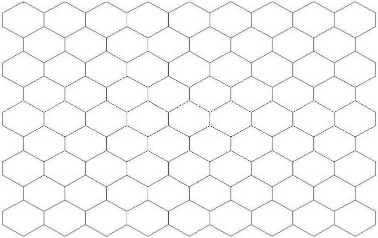
Computational model and validation

- The analytical approach is capable of obtaining equivalent in-plane elastic properties for irregular honeycombs from known random spatial variation of cell angle and material properties of the honeycomb cells.
- The homogenised properties depend on the ratios $h/l, t/l, s/l$ and the angles θ, α, β. In addition, the two Young's moduli and shear modulus also depend on E_s.
- We show results for $h/l = 1.5$ and three values of cell angle θ, namely: 30°, 45° and 60°.
- As the two Young's moduli and shear modulus of low density honeycomb are proportional to $E_s\rho^3$ (Zhu et al., 2001), the non-dimensional results for elastic moduli E_1, E_2, ν_{12}, ν_{21} and G_{12} have been obtained using $\bar{E}_1 = \frac{E_{1eq}}{E_s\rho^3}$, $\bar{E}_2 = \frac{E_{2eq}}{E_s\rho^3}$, $\bar{\nu_{12}} = \nu_{12eq}$, $\bar{\nu_{21}} = \nu_{21eq}$ and $\bar{G_{12}} = \frac{G_{12eq}}{E_s\rho^3}$ respectively, where '$\bar{.}$' represents the non-dimensional elastic modulus and ρ is the relative density of honeycomb (ratio of the planar area of solid to the total planar area of the honeycomb).

Computational model and validation



(a) $\theta = 60^\circ$

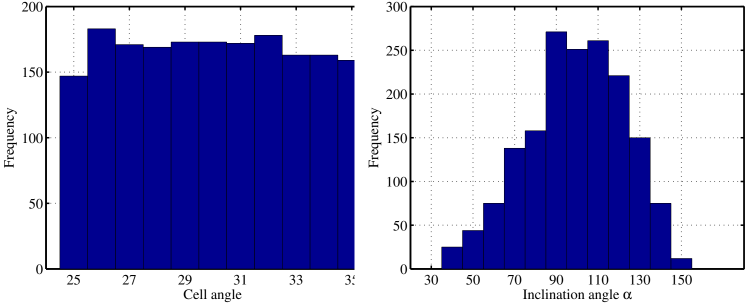


(b) $\theta = 45^\circ$

Figure: Regular honeycomb with different θ values

Computational model and validation

- For the purpose of finding the range of variation in elastic moduli due to spatial uncertainty, cell angles and material properties can be perturbed following a random distribution within specific bounds. We show results for spatial irregularity in the cell angles only.



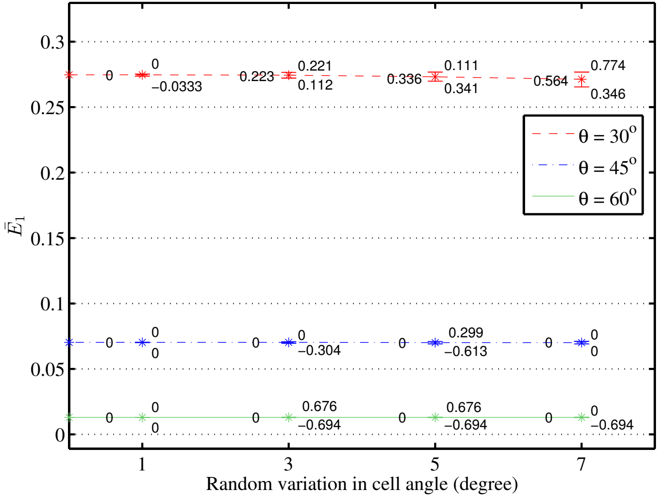
(a) Distribution of cell angle (θ) (b) Distribution of the inclination angle (α)

Figure: Typical statistical distribution of cell angle (θ) and inclination angle α (number of RUCE: 1681)

Computational model and validation

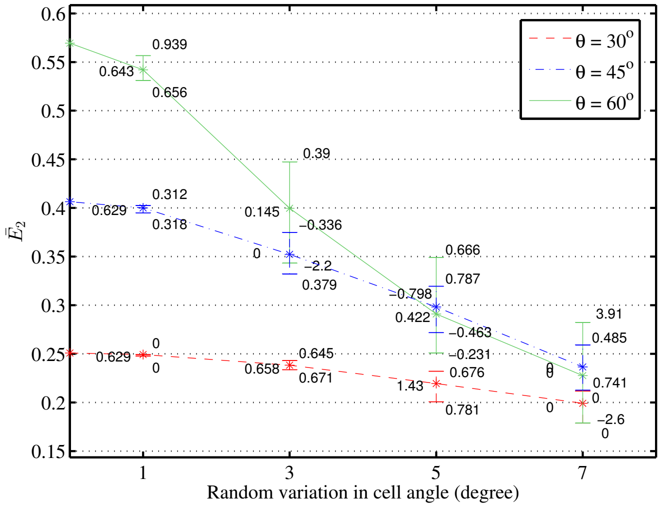
- The maximum, minimum and mean values of non-dimensional in-plane elastic moduli for different degree of spatially random variations in cell angles ($\Delta\theta = 0^\circ, 1^\circ, 3^\circ, 5^\circ, 7^\circ$) are calculated using both direct finite element simulation and the derived closed-form expressions.
- For a particular cell angle θ, results have been obtained using a set of uniformly distributed 1000 random samples in the range of $[\theta - \Delta\theta, \theta + \Delta\theta]$.
- The set of input parameter for a particular sample consists of N number of cell angles in the specified range, where $N(= n \times m)$ is the total number of RUCEs in the entire irregular honeycomb structure. We used 1681 RUCEs (as this was needed for convergence of the deterministic case).
- The quantities h and θ have been considered as the two random input parameters while α, β and l are dependent features.

Longitudinal elastic modulus (E_1)



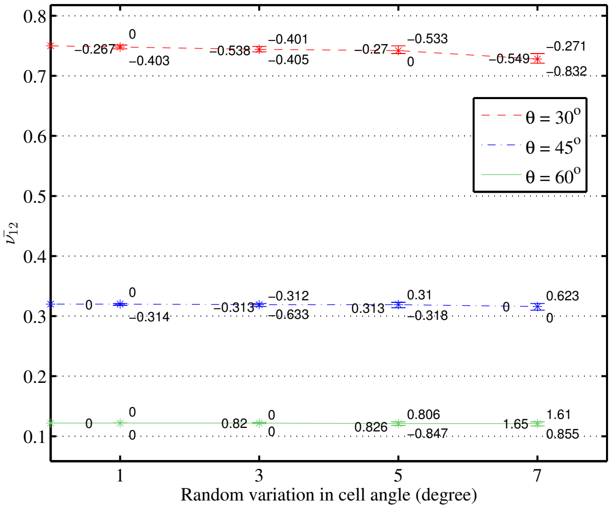
$$h/l = 1.5,\ \bar{E}_1 = E_{1eq}/E_s\rho^3$$

Transverse elastic modulus (E_2)



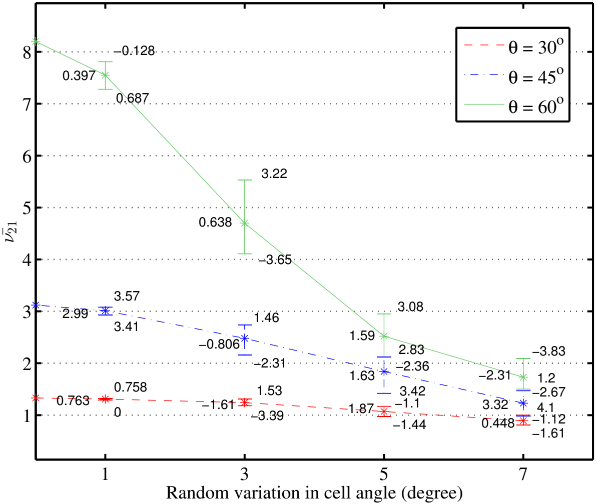
$$h/l = 1.5,\ \bar{E}_2 = E_{2eq}/E_s\rho^3$$

Poisson's ratio ν_{12}



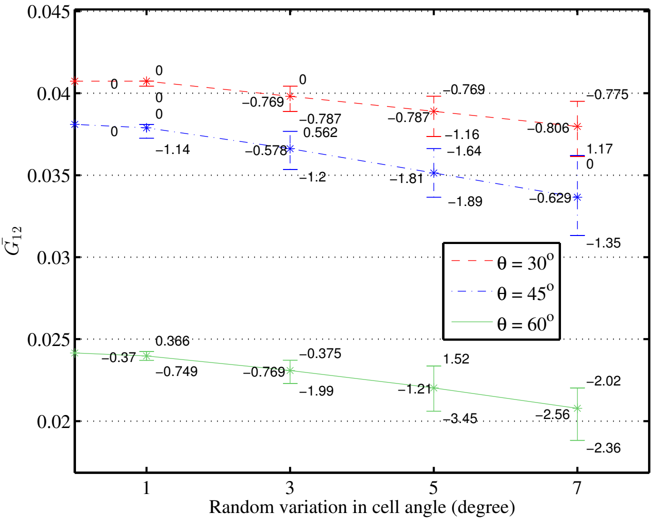
$h/l = 1.5$

Poisson's ratio ν_{21}



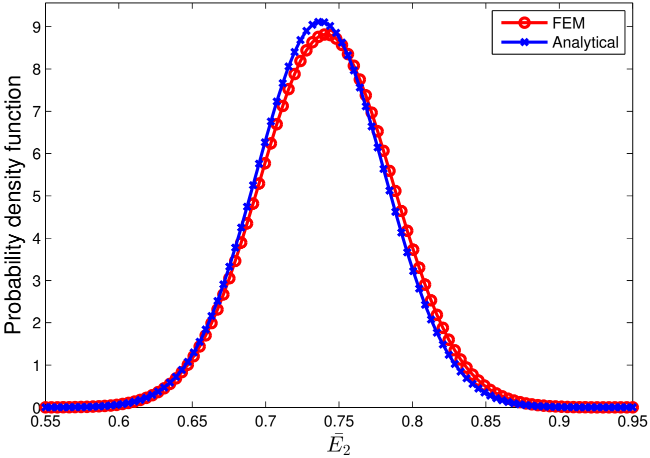
$h/l = 1.5$

Shear modulus (G_{12})



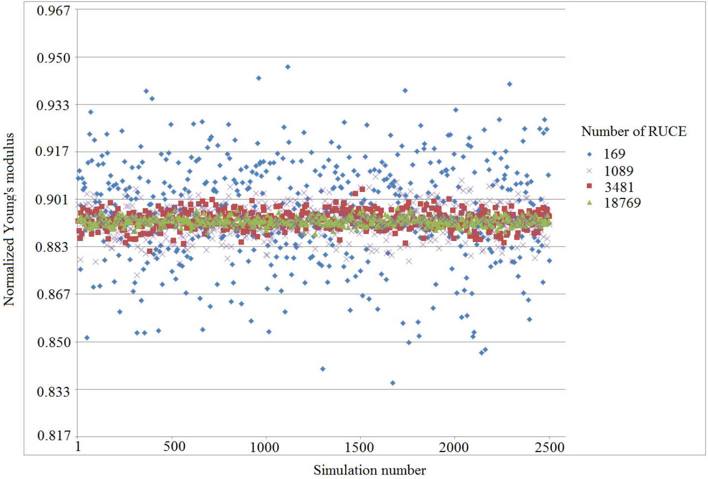
$$h/l = 1.5,\ \bar{G_{12}} = G_{12eq}/E_s\rho^3$$

Probability density function of E_2



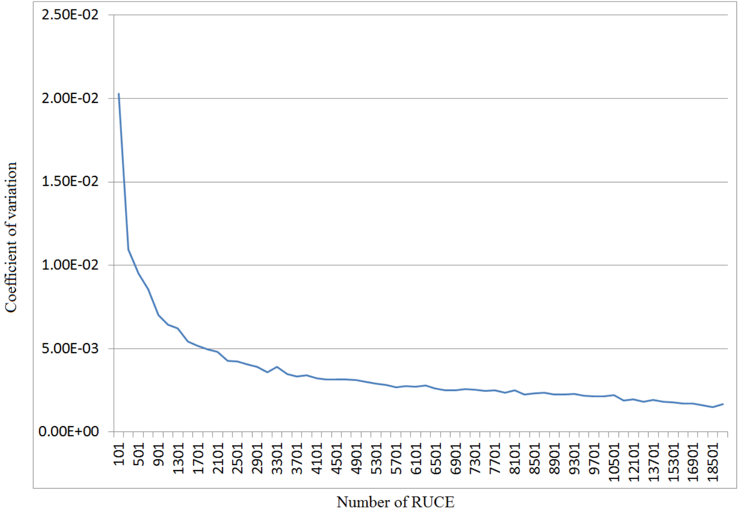
$h/l = 1.5$, $\bar{E}_2 = E_{2eq}/E_{regular}$, $\theta = 45^\circ$, $\Delta\theta = 5^\circ$

Ergodic behaviour of E_2: spread of values



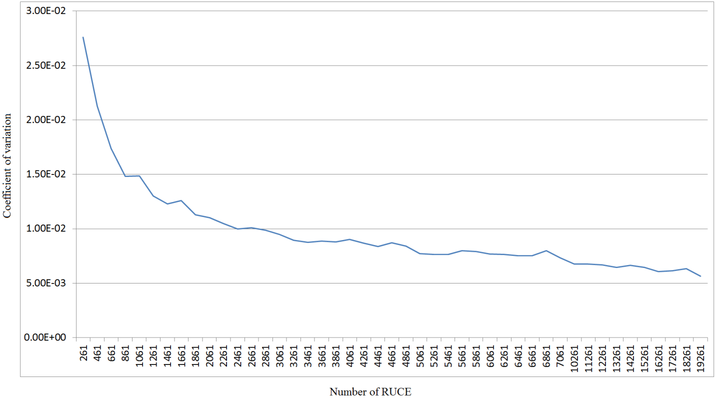
$$h/l = 1.5,\ \bar{E}_2 = E_{2eq}/E_{regular},\ \theta = 45^\circ,\ \Delta\theta = 5^\circ$$

Ergodic behaviour of E_2: coefficient of variation



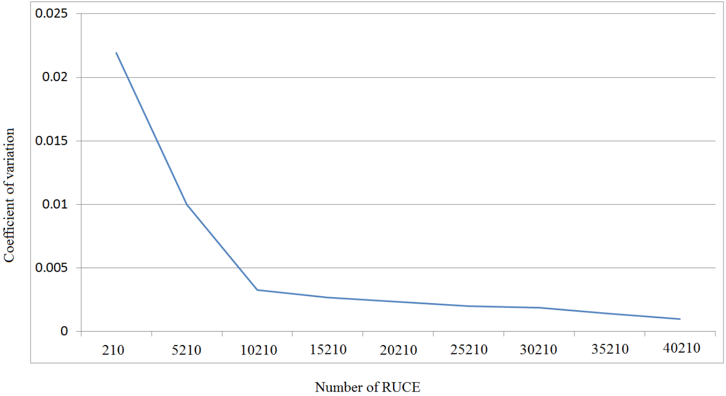
$$h/l = 1.5, \theta = 45^\circ, \Delta\theta = 5^\circ$$

Ergodic behaviour of ν_{21}: coefficient of variation



$$h/l = 1.5,\ \theta = 45^\circ,\ \Delta\theta = 5^\circ$$

Ergodic behaviour of G_{12}: coefficient of variation



$$h/l = 1.5,\ \theta = 45^\circ,\ \Delta\theta = 5^\circ$$

Main observations

- The elastic moduli obtained using the analytical method and by finite element simulation are in good agreement - establishing the validity of the closed-form expressions.
- The number of input random variables (cell angle) increase with the number of cells.
- The variation in E_1 and ν_{12} due to spatially random variations in cell angles is very less, while there is considerable amount of reductions in the values of E_2, ν_{21} and G_{12} with increasing degree of irregularity.
- Longitudinal Young's modulus, transverse Young's modulus and shear modulus are functions of both structural geometry and material properties of the irregular honeycomb (i.e. ratios $h/l, t/l, s/l$ and angles θ, α, β and E_s), while the Poisson's ratios depend only on structural geometry of irregular honeycombs (i.e. ratios $h/l, t/l, s/l$ and angles θ, α, β)
- For large number of random cells ($\approx$1700), we observe the emergence of an effective ergodic behaviour - ensemble statistics become close to single sample "statistics".

Sandwich Panel

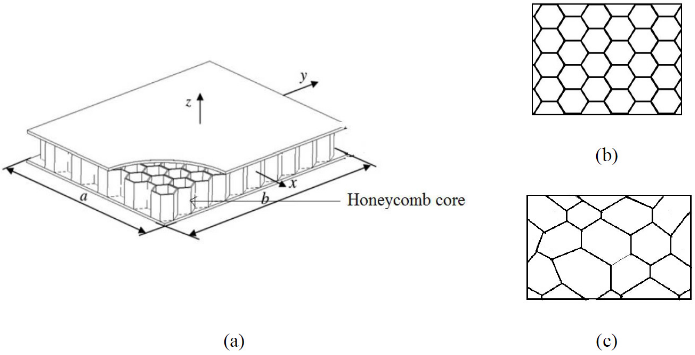
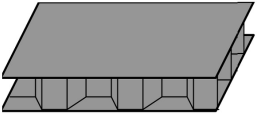


Figure: (a) Sandwich panel (b) Regular honeycomb (c) Irregular honeycomb.

Sandwich panel with irregular honeycomb core



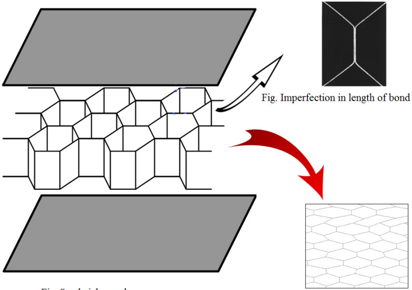
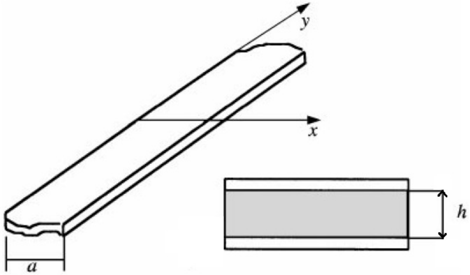


Fig. Imperfection in length of bond lines

Fig. Sandwich panel

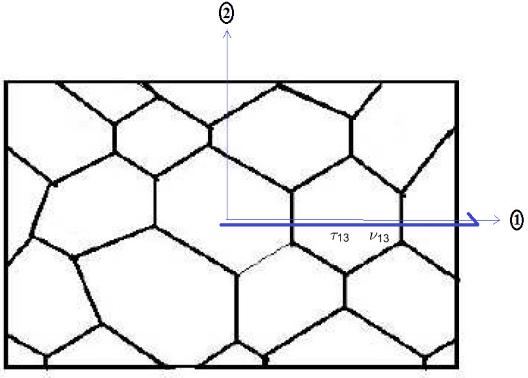
Fig. Randomly spaced over and under expanded cells

Bending vibration of sandwich panels)



- The fundamental natural frequency of sandwich panel having very high length-to-width ratio (Whitney (1987)): $\omega = \frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\sqrt{1 - \frac{S\pi^2}{1 + S\pi^2}}$ where, $S = \frac{D}{G_{13} h a^2}$ and D is the bending stiffness of laminate.
- Thus the fundamental natural frequency depends on G_{13} of the core.

Equivalent G_{13} for irregular honeycomb



- The derivation of G_{13} for irregular honeycomb is described as the out-of-plane shear modulus in the considered problem. However, G_{23} can be derived following similar analogy. Derivation of other out-of-plane shear moduli are straightforward following same way as discussed by Gibson and Ashby (1997) for regular honeycombs.

Minimum potential energy theorem (Gives upper bound of G_{13})

- The strain energy calculated from any postulated set of displacements which are compatible with the external boundary conditions and with themselves, will be a minimum for the exact displacement distribution:
  $\frac{1}{2} G_{13}\nu_{13}^2 V \leq \frac{1}{2}\sum_i G_s {\nu_i}^2 V_i$
  where, G_s is the shear modulus of cell wall material.
- $V(= LBh)$ and $V_i(= l_i th)$ represent the total volume and volume of i^{th} cell wall respectively.
- l_i, t and h are length of i^{th} cell wall, thickness of cell wall and depth of honeycomb core.
- ν_i and ν_{13} represent strain in i^{th} cell wall and global strain respectively. L and B denote overall length and width of entire irregular honeycomb:
  $\nu_i = \nu_{13} cos\theta_i$
  where, $cos\theta_i$ denote the inclination angle of i^{th} cell wall with direction-1.
- From the above equations,
  $\frac{G_{13}}{G_s} \leq \frac{t}{LB}\sum_i l_i cos^2\theta_i$

Minimum complementary energy theorem (Gives lower bound of G_{13})

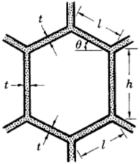
- Among the stress distributions that satisfy equilibrium at each point and are in equilibrium with the external loads, the strain energy is a minimum for the exact stress distribution.
- Expressed as an inequality, for shear in direction-1
  $\frac{1}{2}\frac{\tau_{13}^2}{G_{13}}V \leq \frac{1}{2}\sum_i \frac{\tau_i^2}{G_s}V_i$
- Using the condition of force equilibrium,
  $\tau_{13}LB = \sum_i \tau_i t l_i cos\theta_i$
- From the above two equations, it can be written:
  $\frac{G_{13}}{G_s} \geq \frac{t}{LB}\sum_i l_i cos^2\theta_i$

Expressions for lower and upper bound of G_{13} are noticed to be identical

- Thus considering the lower and upper bound of G_{13}, for irregular honeycomb

$$\frac{G_{13}}{G_s} = \frac{t}{LB} \sum_i l_i cos^2\theta_i$$

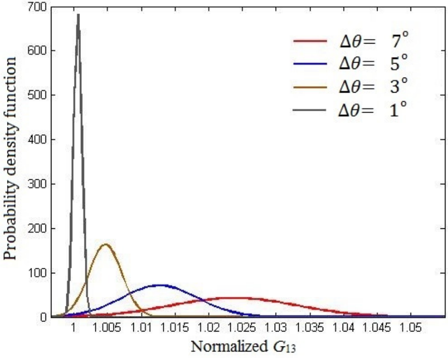
  - Note: The above expression can be reduced to the formula given by Gibson and Ashby (1997) in case of regular hexagonal honeycomb.



For a regular honeycomb as shown in figure:

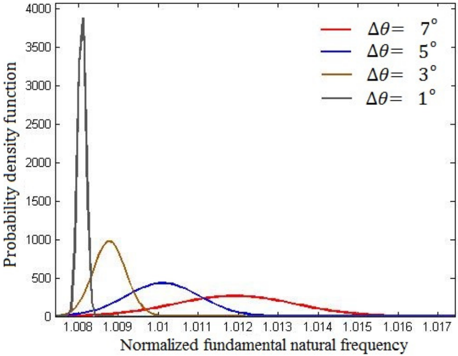
$$\frac{G_{13}}{G_s} = \frac{tcos\theta}{h + lsin\theta}$$

Variation of G_{13} with different degree of irregularity



- G_{13} for irregular honeycomb have been normalized with respect to that of regular honeycomb.

Variation of fundamental natural frequency for the sandwich panel with different degree of irregularity in honeycomb core

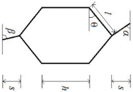


- Fundamental natural frequency for the sandwich panel with irregular honeycomb core have been normalized with respect to that of regular honeycomb core.

Conclusions

- The classical expressions for equivalent in-plane and out of plane elastic properties of regular hexagonal lattice structures have been generalised to consider geometric irregularity.
- Using the principle of basic structural mechanics on an unit cell with a novel homogenisation technique, closed-form expressions have been obtained for E_1, E_2, ν_{12}, ν_{21} and G_{12}.
- On the other hand G_{13} (out of plane) is obtained by simultaneous employment of the minimum potential energy theorem and the minimum complementary energy theorem and subsequent exploitation of two contradictory mathematical inequalities.
- The new results reduce to classical formulae of Gibson and Ashby for the special case of no irregularities.
- Future research will consider more general forms of irregularities.

Conclusions

Irregular honeycomb	Representative unit cell element (RUCE)	Regular honeycomb	Parameter	
			Structural configuration	
$E_{1eq} = \frac{1}{\sum_{j=1}^{n} B_j} \sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} \frac{l_{ij} \cos\theta_{ij}}{E_{1Uij}}} \right) B_j$	$E_{1U} = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l}+\sin\theta)\sin^2\theta}$	$E_{1GA} = E_s \left(\frac{t}{l}\right)^3 \frac{\cos\theta}{(\frac{h}{l}+\sin\theta)\sin^2\theta}$	E_1	In-plane elastic properties
$E_{2eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} E_{2Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j$	$E_{2U} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l}+2\frac{s}{l}+2\sin\theta\right)}{\cos\theta\left(2\cos^2\theta+8\left(\frac{s}{l}\right)^3\left(\frac{\cos^2\alpha}{\sin^3\alpha}+\frac{\cos^2\beta}{\sin^3\beta}\right)+2\left(\frac{s}{l}\right)^2(cot^2\alpha+cot^2\beta)\right)}$	$E_{2GA} = E_s \left(\frac{t}{l}\right)^3 \frac{(\frac{h}{l}+\sin\theta)}{\cos^3\theta}$	E_2	
$\nu_{12eq} = \frac{1}{\sum_{j=1}^{n} B_j} \sum_{j=1}^{n} \left(\frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} \frac{l_{ij} \cos\theta_{ij}}{\nu_{12Uij}}} \right) B_j$	$\nu_{12U} = \frac{2\cos^2\theta}{\left(2\sin\theta+2\frac{s}{l}+\frac{h}{l}\right)\sin\theta}$	$\nu_{12GA} = \frac{\cos^2\theta}{(\frac{h}{l}+\sin\theta)\sin\theta}$	ν_{12}	
$\nu_{21eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} \nu_{21Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j$	$\nu_{21U} = \frac{\sin\theta\left(\frac{h}{l}+2\frac{s}{l}+2\sin\theta\right)}{2\cos^2\theta+8\left(\frac{s}{l}\right)^3\left(\frac{\cos^2\alpha}{\sin^3\alpha}+\frac{\cos^2\beta}{\sin^3\beta}\right)+2\left(\frac{s}{l}\right)^2(cot^2\alpha+cot^2\beta)}$	$\nu_{21GA} = \frac{(\frac{h}{l}+\sin\theta)\sin\theta}{\cos^2\theta}$	ν_{21}	
$G_{12eq} = \frac{1}{\left(\sum_{j=1}^{n} B_j \frac{\sum_{i=1}^{m} l_{ij} \cos\theta_{ij}}{\sum_{i=1}^{m} G_{12Uij} l_{ij} \cos\theta_{ij}} \right)} \sum_{j=1}^{n} B_j$	$G_{12U} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(2\frac{s}{l}+\frac{h}{l}+2\sin\theta\right)}{2\cos\theta\left(2\left(\frac{s}{l}\right)^2+4\left(\frac{s}{l}\right)^3\left(\frac{1}{sin\alpha}+\frac{1}{sin\beta}\right)+\frac{1}{2}\left(\frac{h}{l}\right)^3+\left(\frac{s}{l}+\sin\theta\right)\left(\frac{h}{l}\right)^2\right)}$	$G_{12GA} = E_s \left(\frac{t}{l}\right)^3 \frac{\left(\frac{h}{l}+\sin\theta\right)}{\left(\frac{h}{l}\right)^2(1+2\frac{h}{l})\cos\theta}$	G_{12}	

Some publications

- Mukhopadhyay, T. and Adhikari, S., "Free vibration of sandwich panels with randomly irregular honeycomb core", ASCE Journal of Engineering Mechanics, in press.
- Mukhopadhyay, T. and Adhikari, S., "Equivalent in-plane elastic properties of irregular honeycombs: An analytical approach", International Journal of Solids and Structures, 91[8] (2016), pp. 169-184.
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