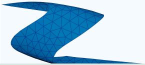
Part 4: Energy harvesting due to random excitations and optimal design

Professor Sondipon Adhikari
FRAeS

Chair of Aerospace Enginering, College of Engineering, Swansea University, Swansea UK
Email: S.Adhikari@swansea.ac.uk, Twitter: @ProfAdhikari
Web: http://engweb.swan.ac.uk/~adhikaris
Google Scholar: http://scholar.google.co.uk/citations?user=tKM35S0AAAAJ

November 2, 2017



Outline of this talk

1. **Introduction**
   - The role of uncertainty
2. **The single-degree-of-freedom coupled model**
   - Energy harvesters without an inductor
   - Energy harvesters with an inductor
3. **Optimal Energy Harvester Under Gaussian Excitation**
   - Stationary random vibration
   - Circuit without an inductor
   - Circuit with an inductor
4. **Stochastic System Parameters**
   - Estimating the mean harvested power by Monte Carlo simulation
   - Optimal parameters: Semi analytical approach
   - Optimal parameters by Monte-Carlo simulations
5. **Summary**

Piezoelectric vibration energy harvesting

- The harvesting of ambient vibration energy for use in powering low energy electronic devices has formed the focus of much recent research.
- Of the published results that focus on the piezoelectric effect as the transduction method, most have focused on harvesting using cantilever beams and on single frequency ambient energy, i.e., resonance based energy harvesting.
- Several authors have proposed methods to optimise the parameters of the system to maximise the harvested energy.
- Some authors have considered energy harvesting under wide band excitation.
- This is crucial to expand the zone of efficiency of the vibration energy harvesters.

Why uncertainty is important for energy harvesting?

- In the context of energy harvesting from ambient vibration, the input excitation may not be always known exactly.
- There may be uncertainties associated with the numerical values considered for various parameters of the harvester. This might arise, for example, due to the difference between the true values and the assumed values.
- If there are several nominally identical energy harvesters to be manufactured, there may be genuine parametric variability within the ensemble.
- Any deviations from the assumed excitation may result an optimally designed harvester to become sub-optimal.

Types of uncertainty

Suppose the set of coupled equations for energy harvesting:

$$\mathcal{L}\{\mathbf{u}(t)\} = \mathbf{f}(t) \tag{1}$$

Uncertainty in the input excitations

- For this case in general $\mathbf{f}(t)$ is a random function of time. Such functions are called random processes.
- $\mathbf{f}(t)$ can be Gaussian/non-Gaussian stationary or non-stationary random processes

Uncertainty in the system

- The operator $\mathcal{L}\{\bullet\}$ is in general a function of parameters $\theta_1, \theta_2, \cdots, \theta_n \in \mathbb{R}$.
- The uncertainty in the system can be characterised by the joint probability density function $p_{\Theta_1,\Theta_2,\cdots,\Theta_n}(\theta_1, \theta_2, \cdots, \theta_n)$.

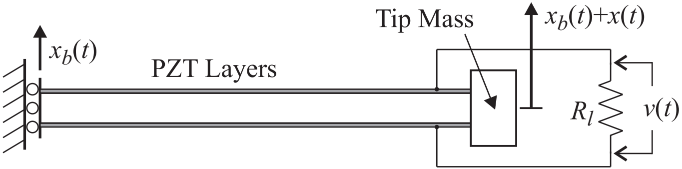
Possible physically realistic cases

Depending on the system and the excitation, several cases are possible:

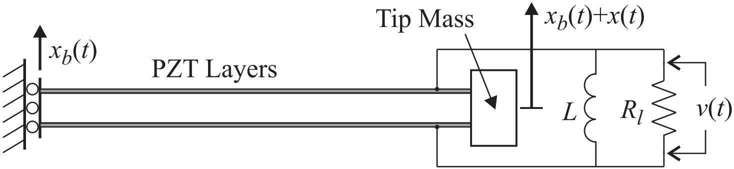
- Linear system excited by harmonic excitation
- Linear system excited by stochastic excitation
- Linear stochastic system excited by harmonic/stochastic excitation
- Nonlinear system excited by harmonic excitation
- Nonlinear system excited by stochastic excitation
- Nonlinear stochastic system excited by harmonic/stochastic excitation
- Multiple degree of freedom vibration energy harvesters

This talk is focused on the application of random vibration theory and random systems theory to various energy harvesting problems.

Energy harvesting circuits - cantilever based



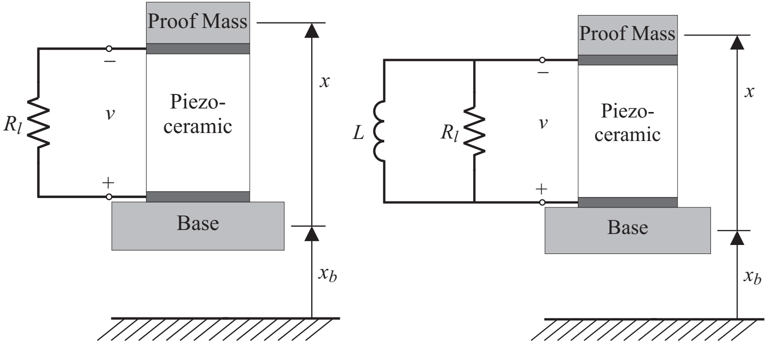
(a) Harvesting circuit without an inductor



(b) Harvesting circuit with an inductor

Figure: Schematic diagrams of piezoelectric energy harvesters with two different harvesting circuits.

Energy harvesting circuits - stack piezo based



Schematic diagrams of stack piezoelectric energy harvesters with two different harvesting circuits. (a) Harvesting circuit without an inductor, (b) Harvesting circuit with an inductor.

The equation of motion

- The coupled electromechanical behaviour of the energy harvester can be expressed by linear ordinary differential equations as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \theta v(t) = -m\ddot{x}_b(t) \tag{2}$$

$$C_p\dot{v}(t) + \frac{1}{R_l}v(t) + \theta\dot{x}(t) = 0 \tag{3}$$

- $x(t)$: displacement of the mass
  m: equivalent mass of the harvester
  k: equivalent stiffness of the harvester
  c: damping of the harvester
  $x_b(t)$: base excitation to the harvester
  θ: electromechanical coupling
  $v(t)$: voltage
  R_l: load resistance
  C_p: capacitance of the piezoelectric layer
  t: time

Frequency domain: non-dimensional form

- Transforming equations (2) and (3) into the frequency domain we obtain and dividing the first equation by m and the second equation by C_p we obtain

$$\left(-\omega^2 + 2\mathrm{i}\omega\zeta\omega_n + \omega_n^2\right) X(\omega) - \frac{\theta}{m} V(\omega) = \omega^2 X_b(\omega) \tag{4}$$

$$\mathrm{i}\omega \frac{\theta}{C_p} X(\omega) + \left(\mathrm{i}\omega + \frac{1}{C_p R_l}\right) V(\omega) = 0 \tag{5}$$

- Here $X(\omega)$, $V(\omega)$ and $X_b(\omega)$ are respectively the Fourier transforms of $x(t)$, $v(t)$ and $x_b(t)$.
- The natural frequency of the harvester, ω_n, and the damping factor, ζ, are defined as

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n} \tag{6}$$

Frequency domain: non-dimensional form

- Dividing the preceding equations by ω_n and writing in matrix form one has

$$\begin{bmatrix} \left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta & -\frac{\theta}{k} \\ \mathrm{i}\Omega\frac{\alpha\theta}{C_p} & (\mathrm{i}\Omega\alpha+1) \end{bmatrix} \begin{Bmatrix} X \\ V \end{Bmatrix} = \begin{Bmatrix} \Omega^2 X_b \\ 0 \end{Bmatrix} \tag{7}$$

- Here the dimensionless frequency and dimensionless time constant are defined as

$$\Omega = \frac{\omega}{\omega_n} \quad \text{and} \quad \alpha = \omega_n C_p R_l \tag{8}$$

- The constant α is the time constant of the first order electrical system, non-dimensionalized using the natural frequency of the mechanical system.

Frequency domain: non-dimensional form

- Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} (\mathrm{i}\Omega\alpha + 1) & \frac{\theta}{k} \\ -\mathrm{i}\Omega\frac{\alpha\theta}{C_p} & \left(1-\Omega^2\right) + 2\mathrm{i}\Omega\zeta \end{bmatrix} \begin{Bmatrix} \Omega^2 X_b \\ 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} (\mathrm{i}\Omega\alpha + 1)\,\Omega^2 X_b/\Delta_1 \\ -\mathrm{i}\Omega^3\frac{\alpha\theta}{C_p}X_b/\Delta_1 \end{Bmatrix} \quad (9)$$

- The determinant of the coefficient matrix is

$$\Delta_1(\mathrm{i}\Omega) = (\mathrm{i}\Omega)^3\alpha + (2\,\zeta\,\alpha+1)\,(\mathrm{i}\Omega)^2 + \left(\alpha + \kappa^2\alpha + 2\,\zeta\right)(\mathrm{i}\Omega) + 1 \quad (10)$$

  This is a cubic equation in $\mathrm{i}\Omega$ leading to to three roots.
- The non-dimensional electromechanical coupling coefficient is

$$\kappa^2 = \frac{\theta^2}{kC_p} \quad (11)$$

The equation of motion

- The coupled electromechanical behaviour of the energy harvester can be expressed by linear ordinary differential equations as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \theta v(t) = f_b(t) \quad (12)$$

$$C_p\ddot{v}(t) + \frac{1}{R_l}\dot{v}(t) + \frac{1}{L}v(t) + \theta\ddot{x}(t) = 0 \quad (13)$$

- Here L is the inductance of the circuit. Note that the mechanical equation is the same as given in equation (2).
- Unlike the previous case, these equations represent two coupled second-order equations and opposed one coupled second-order and one first-order equations.

Frequency domain: non-dimensional form

- Transforming equation (13) into the frequency domain and dividing by $C_p\omega_n^2$ one has

$$-\Omega^2\frac{\theta}{C_p}X+\left(-\Omega^2+\mathrm{i}\Omega\frac{1}{\alpha}+\frac{1}{\beta}\right)V=0 \tag{14}$$

- The second dimensionless constant is defined as

$$\beta=\omega_n^2 LC_p \tag{15}$$

  This is the ratio of the mechanical to electrical natural frequencies.
- Similar to Equation (7), this equation can be written in matrix form with the equation of motion of the mechanical system (4) as

$$\begin{bmatrix}\left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta & -\frac{\theta}{k}\\ -\Omega^2\frac{\alpha\beta\theta}{C_p} & \alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta\end{bmatrix}\begin{Bmatrix}X\\V\end{Bmatrix}=\begin{Bmatrix}\Omega^2X_b\\0\end{Bmatrix} \tag{16}$$

Frequency domain: non-dimensional form

- Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_2} \begin{bmatrix} \alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta & \frac{\theta}{k} \\ \Omega^2 \frac{\alpha\beta\theta}{C_p} & \left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta \end{bmatrix} \begin{Bmatrix} \Omega^2 X_b \\ 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} \left(\alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta\right)\Omega^2 X_b/\Delta_2 \\ \Omega^4 \frac{\alpha\beta\theta}{C_p} X_b/\Delta_2 \end{Bmatrix} \quad (17)$$

- The determinant of the coefficient matrix is

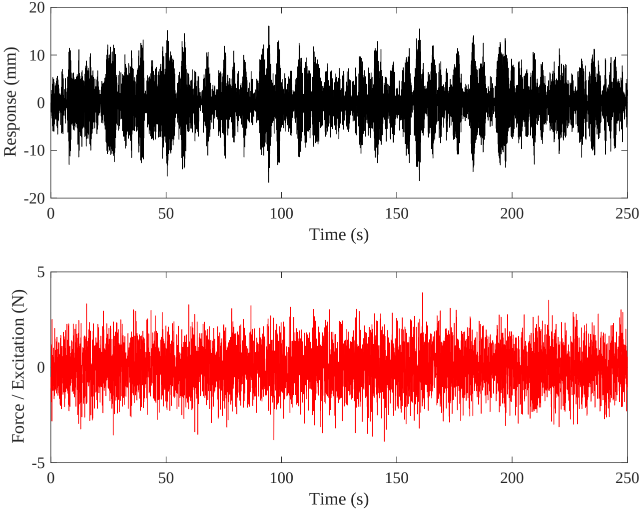
$$\Delta_2(\mathrm{i}\Omega) = (\mathrm{i}\Omega)^4\beta\,\alpha + (2\,\zeta\,\beta\,\alpha+\beta)\,(\mathrm{i}\Omega)^3 + \left(\beta\,\alpha+\alpha+2\,\zeta\,\beta+\kappa^2\beta\,\alpha\right)(\mathrm{i}\Omega)^2 + (\beta+2\,\zeta\,\alpha)\,(\mathrm{i}\Omega)+\alpha \quad (18)$$

This is a quartic equation in $\mathrm{i}\Omega$ leading to to four roots.

Stationary random vibration

- We consider that the base excitation $x_b(t)$ is a random process.
- It is assumed that $x_b(t)$ is a weakly stationary, Gaussian, broadband random process.
- Mechanical systems driven by this type of excitation have been discussed by Lin [1], Nigam [2], Bolotin [3], Roberts and Spanos [4] and Newland [5] within the scope of random vibration theory.
- To obtain the samples of the random response quantities such as the displacement of the mass $x(t)$ and the voltage $v(t)$, one needs to solve the coupled stochastic differential equations (2) and (3) or (2) and (13).

Stationary random vibration



Input force and output response of a linear harvester with an inductor.

Stationary random vibration

- Analytical results developed within the theory of random vibration allows us to bypass numerical solutions because we are interested in the average values of the output random processes. Here we extend the available results to the energy harvester.
- Since $x_b(t)$ is a weakly stationary random process, its autocorrelation function depends only on the difference in the time instants, and thus

$$\mathrm{E}\left[x_b(\tau_1)x_b(\tau_2)\right] = R_{x_b x_b}(\tau_1 - \tau_2) \tag{19}$$

- This autocorrelation function can be expressed as the inverse Fourier transform of the spectral density $\Phi_{x_b x_b}(\omega)$ as

$$R_{x_b x_b}(\tau_1 - \tau_2) = \int_{-\infty}^{\infty} \Phi_{x_b x_b}(\omega) \exp[\mathrm{i}\omega(\tau_1 - \tau_2)]\mathrm{d}\omega \tag{20}$$

Stationary random vibration

- We are interested in the average harvested power given by

$$\mathrm{E}\left[P(t)\right] = \mathrm{E}\left[\frac{v^2(t)}{R_l}\right] = \frac{\mathrm{E}\left[v^2(t)\right]}{R_l} \tag{21}$$

- For a damped linear system of the form $V(\omega) = H(\omega)X_b(\omega)$, it can be shown that [1, 2] the spectral density of V is related to the spectral density of X_b by

$$\Phi_{VV}(\omega) = |H(\omega)|^2 \Phi_{x_b x_b}(\omega) \tag{22}$$

- Thus, for large t, we obtain

$$\mathrm{E}\left[v^2(t)\right] = R_{vv}(0) = \int_{-\infty}^{\infty} |H(\omega)|^2 \Phi_{x_b x_b}(\omega)\, \mathrm{d}\omega \tag{23}$$

- This expression will be used to obtain the average power for the two cases considered. We assume that the base acceleration $\ddot{x}_b(t)$ is Gaussian white noise so that its spectral density is constant with respect to the frequency.

Stationary random vibration

- The calculation of the integral on the right-hand side of Equation (23) in general requires the calculation of integrals of the following form

$$I_n = \int_{-\infty}^{\infty} \frac{\Xi_n(\omega)\,\mathrm{d}\omega}{\Lambda_n(\omega)\Lambda_n^*(\omega)} \tag{24}$$

- The polynomials have the form

$$\Xi_n(\omega) = b_{n-1}\omega^{2n-2} + b_{n-2}\omega^{2n-4} + \cdots + b_0 \tag{25}$$

$$\Lambda_n(\omega) = a_n(\mathrm{i}\omega)^n + a_{n-1}(\mathrm{i}\omega)^{n-1} + \cdots + a_0 \tag{26}$$

- Following Roberts and Spanos [4] this integral can be evaluated as

$$I_n = \frac{\pi}{a_n}\frac{\det\left[\mathbf{D}_n\right]}{\det\left[\mathbf{N}_n\right]} \tag{27}$$

Stationary random vibration

- Here the $m \times m$ matrices are defined as

$$\mathbf{D}_n = \begin{bmatrix} b_{n-1} & b_{n-2} & & \cdots & & & b_0 \\ -a_n & a_{n-2} & -a_{n-4} & a_{n-6} & \cdots & 0 & \cdots \\ 0 & -a_{n-1} & a_{n-3} & -a_{n-5} & \cdots & 0 & \cdots \\ 0 & a_n & -a_{n-2} & a_{n-4} & \cdots & 0 & \cdots \\ 0 & \cdots & & & \cdots & 0 & \cdots \\ 0 & 0 & & & \cdots & -a_2 & a_0 \end{bmatrix} \tag{28}$$

and

$$\mathbf{N}_n = \begin{bmatrix} a_{n-1} & -a_{n-3} & a_{n-5} & -a_{n-7} & & & \\ -a_n & a_{n-2} & -a_{n-4} & a_{n-6} & \cdots & 0 & \cdots \\ 0 & -a_{n-1} & a_{n-3} & -a_{n-5} & \cdots & 0 & \cdots \\ 0 & a_n & -a_{n-2} & a_{n-4} & \cdots & 0 & \cdots \\ 0 & \cdots & & & \cdots & 0 & \cdots \\ 0 & 0 & & & \cdots & -a_2 & a_0 \end{bmatrix} \tag{29}$$

Circuit without an inductor

Mean power

The average harvested power due to the white-noise base acceleration with a circuit without an inductor can be obtained as

$$\mathrm{E}\left[\widetilde{P}\right] = \mathrm{E}\left[\frac{|V|^2}{(R_l \omega^4 \Phi_{x_b x_b})}\right] = \frac{\pi\, m\, \alpha\, \kappa^2}{(2\,\zeta\,\alpha^2+\alpha)\,\kappa^2 + 4\,\zeta^2\alpha + (2\,\alpha^2+2)\,\zeta}$$

- From Equation (9) we obtain the voltage in the frequency domain as

$$V = \frac{-\mathrm{i}\Omega^3 \frac{\alpha\theta}{C_p}}{\Delta_1(\mathrm{i}\Omega)} X_b \tag{30}$$

- We are interested in the mean of the normalized harvested power when the base acceleration is Gaussian white noise, that is $|V|^2/(R_l\omega^4\Phi_{x_b x_b})$.

Mean power

- The spectral density of the acceleration $\omega^4\Phi_{x_bx_b}$ and is assumed to be constant. After some algebra, from Equation (30), the normalized power is

$$\widetilde{P} = \frac{|V|^2}{(R_l\omega^4\Phi_{x_bx_b})} = \frac{k\alpha\kappa^2}{\omega_n^3}\frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)} \tag{31}$$

- Using linear stationary random vibration theory, the average normalized power can be obtained as

$$\mathrm{E}\left[\widetilde{P}\right] = \mathrm{E}\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_bx_b})}\right] = \frac{k\alpha\kappa^2}{\omega_n^3}\int_{-\infty}^{\infty}\frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)}\,\mathrm{d}\omega \tag{32}$$

- From Equation (10) observe that $\Delta_1(\mathrm{i}\Omega)$ is a third order polynomial in $(\mathrm{i}\Omega)$. Noting that $\mathrm{d}\omega = \omega_n \mathrm{d}\Omega$ and from Equation (10), the average harvested power can be obtained from Equation (32) as

$$\mathrm{E}\left[\widetilde{P}\right] = \mathrm{E}\left[\frac{|V|^2}{(R_l\omega^4\Phi_{x_bx_b})}\right] = m\alpha\kappa^2 I^{(1)} \tag{33}$$

Mean power

- $$I^{(1)} = \int_{-\infty}^{\infty} \frac{\Omega^2}{\Delta_1(\mathrm{i}\Omega)\Delta_1^*(\mathrm{i}\Omega)} \, \mathrm{d}\Omega \tag{34}$$
- After some algebra, this integral can be evaluated as

$$I^{(1)} = \frac{\pi}{\alpha} \frac{\det \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & \alpha + \kappa^2\alpha + 2\,\zeta & 0 \\ 0 & -2\,\zeta\,\alpha - 1 & 1 \end{bmatrix}}{\det \begin{bmatrix} 2\,\zeta\,\alpha + 1 & -1 & 0 \\ -\alpha & \alpha + \kappa^2\alpha + 2\,\zeta & 0 \\ 0 & -2\,\zeta\,\alpha - 1 & 1 \end{bmatrix}} \tag{35}$$

Combining this with Equation (33) we obtain the average harvested power due to white-noise base acceleration.

Optimal mean power

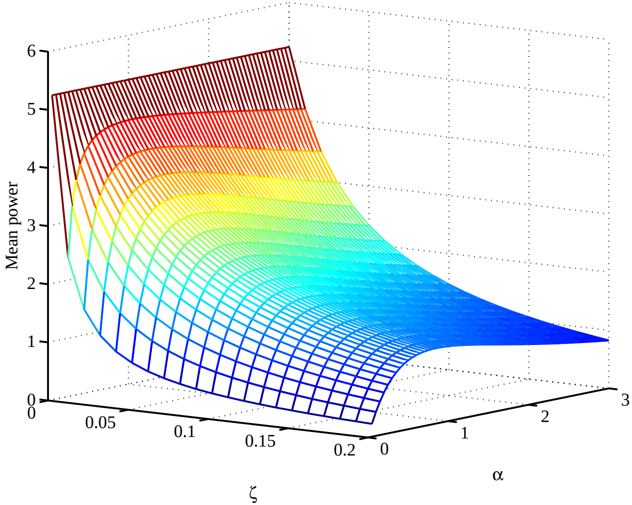
- Since α and κ^2 are positive the average harvested power is monotonically decreasing with damping ratio ζ. Thus the mechanical damping in the harvester should be minimised.
- For fixed α and ζ the average harvested power is monotonically increasing with the coupling coefficient κ^2, and hence the electromechanical coupling should be as large as possible.
- Maximizing the average power with respect to α gives the condition

$$\alpha^2 \left(1 + \kappa^2\right) = 1 \tag{36}$$

or in terms of physical quantities

$$R_l^2 C_p \left(kC_p + \theta^2\right) = m \tag{37}$$

Normalised mean power: numerical illustration



The normalised mean power of a harvester without an inductor as a function of α and ζ, with $\kappa = 0.6$.

Circuit with an inductor

Mean power

The average harvested power due to the white-noise base acceleration with a circuit with an inductor can be obtained as

$$\mathrm{E}\left[\widetilde{P}\right] = \frac{m\alpha\beta\kappa^2\pi\left(\beta+2\alpha\zeta\right)}{\beta\left(\beta+2\alpha\zeta\right)\left(1+2\alpha\zeta\right)\left(\alpha\kappa^2+2\zeta\right)+2\alpha^2\zeta\left(\beta-1\right)^2}$$

Mean power

- Here
$$I^{(2)} = \int_{-\infty}^{\infty} \frac{\Omega^4}{\Delta_2(\mathrm{i}\Omega)\Delta_2^*(\mathrm{i}\Omega)} \, \mathrm{d}\Omega. \tag{38}$$
- Using the expression of $\Delta_2(\mathrm{i}\Omega)$ in Equation (18) and comparing $I^{(2)}$ with the general integral in Equation (24) we have

$$\begin{aligned} n = 4, b_3 = 0, b_2 = 1, b_1 = 0, b_0 = 0, a_4 = \beta\alpha, a_3 = (2\,\zeta\,\beta\,\alpha + \beta) \\ a_2 = \left(\beta\,\alpha + \alpha + 2\,\zeta\,\beta + \kappa^2\beta\,\alpha\right), a_1 = (\beta + 2\,\zeta\,\alpha)\,, a_0 = \alpha \end{aligned} \tag{39}$$

Mean power

- Using Equation (27), the integral can be evaluated as

$$I^{(2)} = \frac{\pi}{\beta\alpha} \frac{\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\beta\,\alpha & \beta\,\alpha + \alpha + 2\,\zeta\,\beta + \kappa^2\beta\,\alpha & -\alpha & 0 \\ 0 & -2\,\zeta\,\beta\,\alpha - \beta & \beta + 2\,\zeta\,\alpha & 0 \\ 0 & -\beta\,\alpha & \beta\,\alpha + \alpha + 2\,\zeta\,\beta + \kappa^2\beta\,\alpha & \alpha \end{bmatrix}}{\det \begin{bmatrix} 2\,\zeta\,\beta\,\alpha + \beta & -\beta - 2\,\zeta\,\alpha & 0 & 0 \\ -\beta\,\alpha & \beta\,\alpha + \alpha + 2\,\zeta\,\beta + \kappa^2\beta\,\alpha & -\alpha & 0 \\ 0 & -2\,\zeta\,\beta\,\alpha - \beta & \beta + 2\,\zeta\,\alpha & 0 \\ 0 & -\beta\,\alpha & \beta\,\alpha + \alpha + 2\,\zeta\,\beta + \kappa^2\beta\,\alpha & \alpha \end{bmatrix}} \tag{40}$$

Mean power

- Combining this with Equation (33) we finally obtain the average normalized harvested power as

$$
\begin{aligned}
\mathrm{E}\left[\widetilde{P}\right] &= \mathrm{E}\left[\frac{|V|^2}{(R_l \omega^4 \Phi_{x_b x_b})}\right] \\
&= m\alpha\beta\kappa^2\pi\left(\beta+2\alpha\zeta\right) / \left[\left(4\beta\alpha^3\zeta^2 + 2\beta\alpha^2\left(\beta+1\right)\zeta + \beta^2\alpha\right)\kappa^2 \right. \\
&\qquad \left. + 8\beta\alpha^2\zeta^3 + 4\beta\alpha\left(\beta+1\right)\zeta^2 + 2\left(\beta^2\alpha^2 + \beta^2 - 2\beta\alpha^2 + \alpha^2\right)\zeta\right] \\
&= \frac{m\alpha\beta\kappa^2\pi\left(\beta+2\alpha\zeta\right)}{\beta\left(\beta+2\alpha\zeta\right)\left(1+2\alpha\zeta\right)\left(\alpha\kappa^2+2\zeta\right) + 2\alpha^2\zeta\left(\beta-1\right)^2}
\end{aligned}
\tag{41}
$$

- This is the complete closed-form expression of the normalised harvested power under Gaussian white noise base acceleration.
- Since α, β and κ^2 are positive the average harvested power is monotonically decreasing with damping ratio ζ.
- The mechanical damping in the harvester should be minimised.

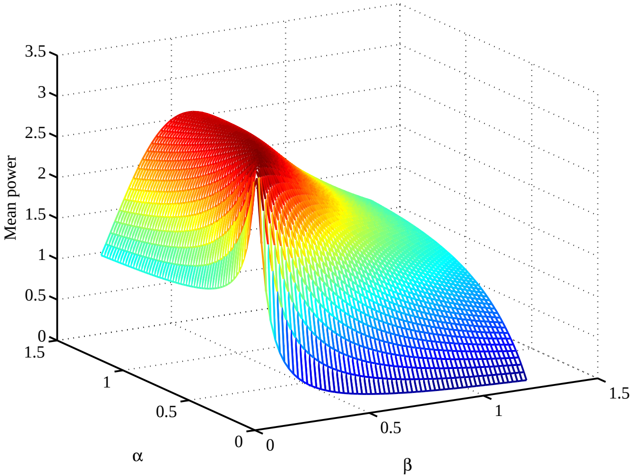
Mean power

- For fixed α, β and ζ the average harvested power is monotonically increasing with the coupling coefficient κ^2, and hence the electromechanical coupling should be as large as possible.
- We can also determine optimum values for α and β. Dividing both the numerator and denominator of the last expression in Equation (41) by $\beta\left(\beta+2\alpha\zeta\right)$ shows that the optimum value of β for all values of the other parameters is $\beta = 1$.
- This value of β implies that $\omega_n^2 L C_p = 1$, and thus the mechanical and electrical natural frequencies are equal. With $\beta = 1$ the average normalised harvested power is

$$\mathrm{E}\left[\widetilde{P}\right] = \frac{m\alpha\kappa^2\pi}{\left(1+2\alpha\zeta\right)\left(\alpha\kappa^2+2\zeta\right)}. \tag{42}$$

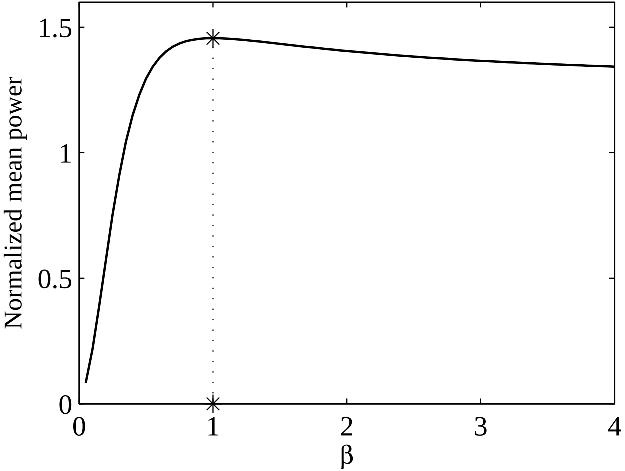
If κ and ζ are fixed then the maximum power with respect to α is obtained when $\alpha = 1/\kappa$.

Normalised mean power: numerical illustration



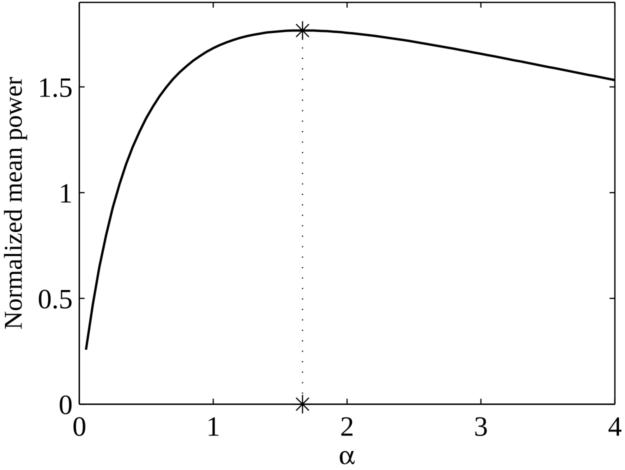
The normalized mean power of a harvester with an inductor as a function of α and β, with $\zeta = 0.1$ and $\kappa = 0.6$.

Optimal parameter selection



The normalized mean power of a harvester with an inductor as a function of β for $\alpha = 0.6$, $\zeta = 0.1$ and $\kappa = 0.6$. The * corresponds to the optimal value of $\beta (= 1)$ for the maximum mean harvested power.

Optimal parameter selection



The normalized mean power of a harvester with an inductor as a function of α for $\beta = 1$, $\zeta = 0.1$ and $\kappa = 0.6$. The * corresponds to the optimal value of $\alpha (= 1.667)$ for the maximum mean harvested power.

Stochastic system parameters

- Energy harvesting devices are expected to be produced in bulk quantities
- It is expected to have some parametric variability across the 'samples'
- How can we take this into account and optimally design the parameters?

The natural frequency of the harvester, ω_n, and the damping factor, ζ_n, are assumed to be random in nature and are defined as

$$\omega_n = \bar{\omega}_n \Psi_\omega \tag{43}$$

$$\zeta = \bar{\zeta} \Psi_\zeta \tag{44}$$

where Ψ_ω and Ψ_ζ are the random parts of the natural frequency and damping coefficient. $\bar{\omega}_n$ and $\bar{\zeta}$ are the mean values of the natural frequency and damping coefficient.

Mean harvested power: Harmonic excitation

The average (mean) normalized power can be obtained as

$$\begin{aligned}\mathrm{E}[P] &= \mathrm{E}\left[\frac{|V|^2}{(R_l\omega^4 X_b^2)}\right] \\ &= \frac{\bar{k}\alpha\kappa^2\Omega^2}{\bar{\omega}_n^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_{\Psi_\omega}(x_1)f_{\Psi_\zeta}(x_2)}{\Delta_1(\mathrm{i}\Omega, x_1, x_2)\Delta_1^*(\mathrm{i}\Omega, x_1, x_2)}\,\mathrm{d}x_1\mathrm{d}x_2 \quad (45)\end{aligned}$$

where

$$\begin{aligned}\Delta_1(\mathrm{i}\Omega, \Psi_\omega, \Psi_\zeta) = (\mathrm{i}\Omega)^3\alpha + \left(2\bar{\zeta}\alpha\Psi_\omega\Psi_\zeta + 1\right)(\mathrm{i}\Omega)^2 + \\ \left(\alpha\Psi_\omega^2 + \kappa^2\alpha + 2\bar{\zeta}\Psi_\omega\Psi_\zeta\right)(\mathrm{i}\Omega) + \Psi_\omega^2 \quad (46)\end{aligned}$$

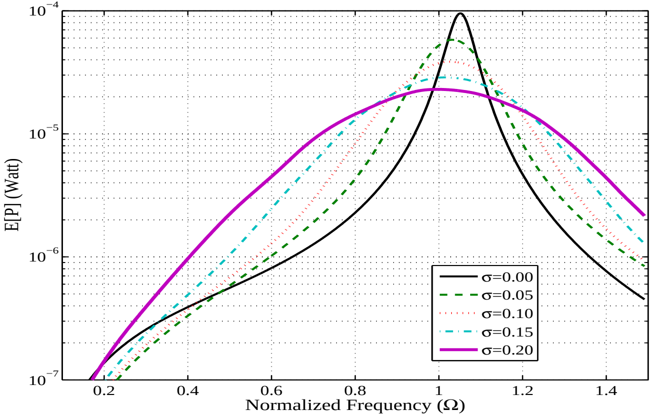
The probability density functions (pdf) of Ψ_ω and Ψ_ζ are denoted by $f_{\Psi_\omega}(x)$ and $f_{\Psi_\zeta}(x)$ respectively.

Parameter values used in the simulation

Table: Parameter values used in the simulation

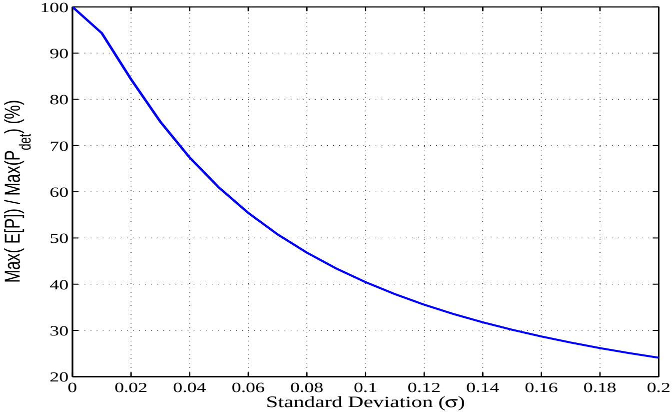
Parameter	Value	Unit
m	9.12×10^{-3}	kg
$\bar{k}$	4.1×10^{3}	N/m
$\bar{c}$	0.218	Ns/m
α	0.8649	–
R_l	3×10^{4}	Ohm
κ^2	0.1185	–
C_p	4.3×10^{-8}	F
θ	-4.57×10^{-3}	N/V

The mean power



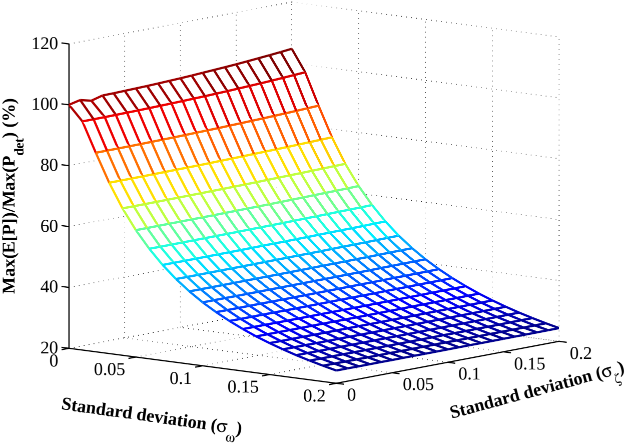
The mean power for various values of standard deviation in natural frequency with $\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \Psi_\zeta = 1, \alpha = 0.8649, \kappa^2 = 0.1185$.

The mean power



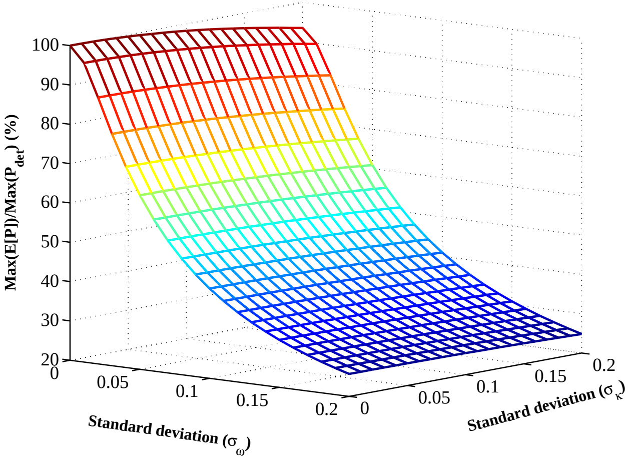
The mean harvested power for various values of standard deviation of the natural frequency, normalised by the deterministic power ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \Psi_\zeta = 1, \alpha = 0.8649, \kappa^2 = 0.1185$).

The mean harvested power



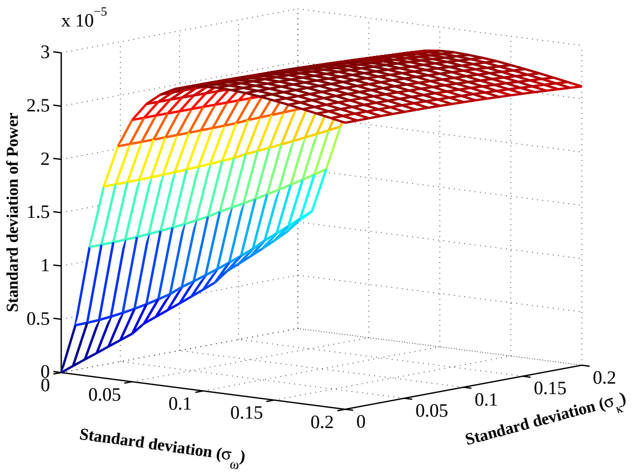
The mean harvested power for various values of standard deviation of the natural frequency (σ_ω) and damping coefficient (σ_ζ), normalised by the deterministic power ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \bar{\zeta} = 0.0178, \alpha = 0.8649, \kappa^2 = 0.1185$).

The mean harvested power



The mean harvested power for various values of standard deviation of the natural frequency (σ_ω) and non-dimensional coupling coefficient (σ_κ), normalised by the deterministic power with $\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \bar{\kappa} = 0.3342, \Psi_\zeta = 1, \alpha = 0.8649$.

Standard deviation of harvested power



The standard deviation of harvested power for various values of standard deviation of the natural frequency (σ_ω) and non-dimensional coupling coefficient (σ_κ), normalised by the deterministic power with $\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \bar{\kappa} = 0.3342, \Psi_\zeta = 1, \alpha = 0.8649$.

Optimal parameter selection

The optimal value of α:

$$\alpha_{\rm opt}^2 \approx \frac{(c_1 + c_2\sigma^2 + 3c_3\sigma^4)}{(c_4 + c_5\sigma^2 + 3c_6\sigma^4)} \tag{47}$$

where

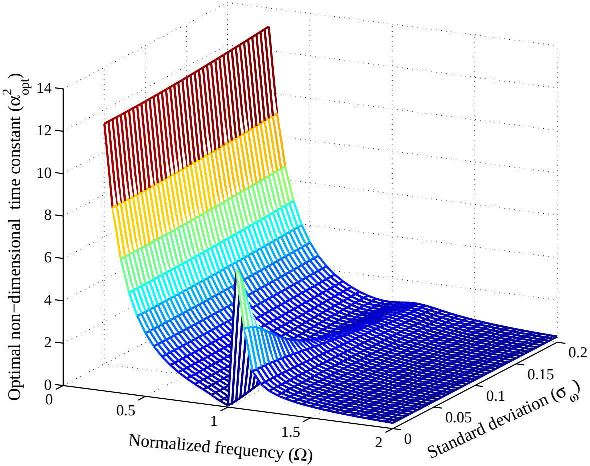
$$c_1 = 1 + \left(4\bar{\zeta}^2 - 2\right)\Omega^2 + \Omega^4, c_2 = 6 + \left(4\bar{\zeta}^2 - 2\right)\Omega^2, c_3 = 1, \tag{48}$$

$$c_4 = \left(1 + 2\kappa^2 + \kappa^4\right)\Omega^2 + \left(4\bar{\zeta}^2 - 2 - 2\kappa^2\right)\Omega^4 + \Omega^6, \tag{49}$$

$$c_5 = \left(2\kappa^2 + 6\right)\Omega^2 + \left(4\bar{\zeta}^2 - 2\right)\Omega^4, \qquad c_6 = \Omega^2, \tag{50}$$

and σ is the standard deviation in natural frequency.

Optimal non-dimensional time constant



The optimal non-dimensional time constant $\alpha_{\rm opt}$ for various standard deviations of the natural frequency under a broad band excitation

Optimal parameter selection

The optimal value of κ:

$$\kappa_{\text{opt}}^2 \approx \frac{1}{(\alpha\ \Omega)} \sqrt{(d_1 + d_2\sigma^2 + d_3\sigma^4)} \tag{51}$$

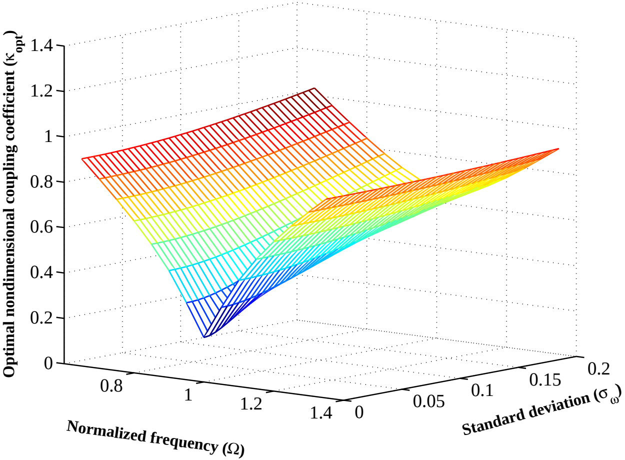
where

$$d_1 = 1 + \left(4\bar{\zeta}^2 + \alpha^2 - 2\right)\Omega^2 + \left(4\bar{\zeta}^2\alpha^2 - 2\alpha^2 + 1\right)\Omega^4 + \alpha^2\Omega^6 \tag{52}$$

$$d_2 = 6 + \left(4\bar{\zeta}^2 + 6\alpha^2 - 2\right)\Omega^2 + \left(4\bar{\zeta}^2\alpha^2 - 2\alpha^2\right)\Omega^4 \tag{53}$$

$$d_3 = 3 + 3\alpha^2\Omega^2 \tag{54}$$

Optimal non-dimensional coupling coefficient

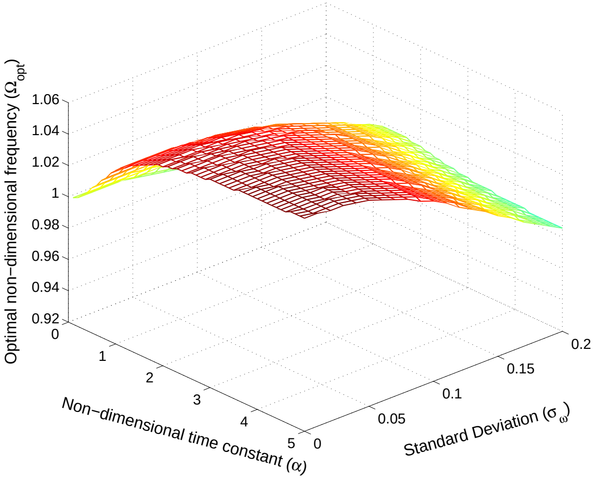


The optimal non-dimensional coupling coefficient $\kappa_{\rm opt}$ for various standard deviations of the natural frequency under a broad band excitation ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \alpha = 0.8649$).

Monte-Carlo simulations

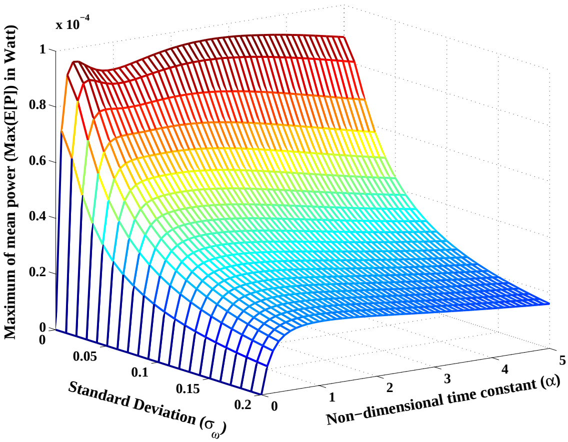
- The simulation is performed using 5000 sample realisations, with the standard deviation of natural frequency between 0 and 20% at an interval of 1%.
- The non-dimensional time constant α is varied from 0 to 5 with an interval of 0.1.
- For the non-dimensional coupling coefficient, the simulation range considered is from 0 to 1 at an interval of 0.05. The optimal values of the parameters lie well within conventional ranges. For the various simulations, any constant parameters used are given in Table 1.

The optimal non-dimensional frequency



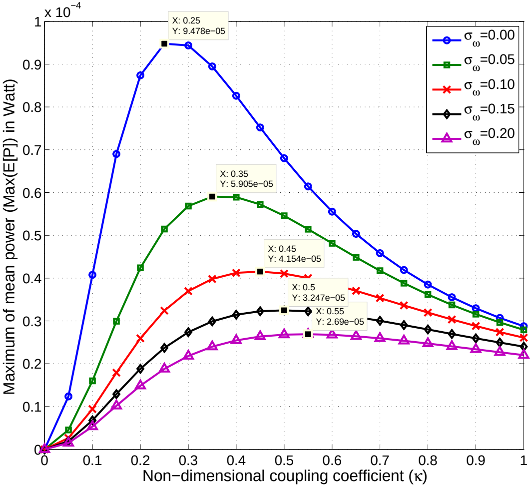
The optimal non-dimensional frequency (Ω_{opt}) obtained using MCS ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \kappa^2 = 0.1185$).

The maximum of the mean non-dimensional power



The maximum of the mean non-dimensional power ($\max(\mathrm{E}\left[P\right])$) obtained using MCS ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \kappa^2 = 0.1185$).

The mean non-dimensional power



The mean non-dimensional power ($\max(\mathrm{E}[P])$) obtained using MCS ($\bar{\omega}_n = 670.5\,\mathrm{rad/s}, \alpha = 0.8649$).

Main observations

- The mean harvested power decreases abruptly with uncertainty in the natural frequency, as shown by the approximate analytical results and the Monte-Carlo simulation studies
- The harvested power varies only slightly due to uncertainty in the damping (up to a standard deviation of 20%)
- The sharp peak in the optimal non-dimensional time constant gradually vanishes with increasing standard deviation of the natural frequency
- The harvested power decreases with increase in uncertainty in coupling coefficient at low uncertainty in natural frequency. As the uncertainty in natural frequency increases the effect of uncertainty in coupling coefficient decreases
- The value of the non-dimensional coupling coefficient at which the mean harvested power is maximum increases with increasing standard deviation of the natural frequency
- The optimal value for the non-dimensional frequency decreases with increasing standard deviation of the natural frequency.

Summary

- Vibration energy based piezoelectric energy harvesters are expected to operate under a wide range of ambient environments. This talk considers energy harvesting of such systems under broadband random excitations.
- Specifically, analytical expressions of the normalised mean harvested power due to stationary Gaussian white noise base excitation has been derived.
- Two cases, namely the harvesting circuit with and without an inductor, have been considered.
- It was observed that in order to maximise the mean of the harvested power (a) the mechanical damping in the harvester should be minimised, and (b) the electromechanical coupling should be as large as possible.

Further reading

[1] Y. K. Lin, Probabilistic Thoery of Strcutural Dynamics, McGraw-Hill Inc, Ny, USA, 1967.
[2] N. C. Nigam, Introduction to Random Vibration, The MIT Press, Cambridge, Massachusetts, 1983.
[3] V. V. Bolotin, Random vibration of elastic systems, Martinus and Nijhoff Publishers, The Hague, 1984.
[4] J. B. Roberts, P. D. Spanos, Random Vibration and Statistical Linearization, John Wiley and Sons Ltd, Chichester, England, 1990.
[5] D. E. Newland, Mechanical Vibration Analysis and Computation, Longman, Harlow and John Wiley, New York, 1989.