Part 2: Piezoelectric Energy harvesting due to harmonic excitations

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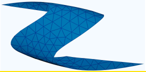


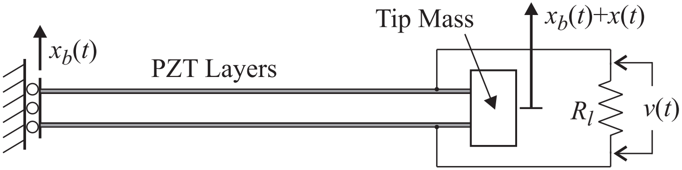
Outline of this talk

1. **Introduction**
2. **The single-degree-of-freedom coupled model**
3. **Energy harvesters without an inductor**
   - Time-domain and state-space equation
   - Response in the frequency domain
4. **Energy harvesters with an inductor**
   - Time-domain and state space equation
   - Response in the frequency domain
5. **Summary**

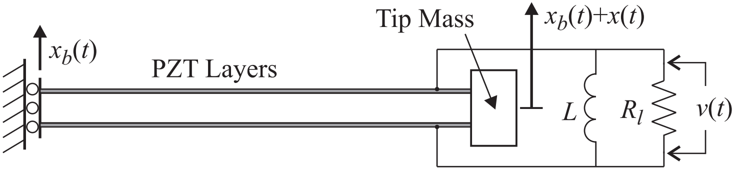
Coupled SDOF model

- The dynamics of a cantilever beam with a piezoelectric patch and tip mass can be expressed by an equivalent single-degree-of-freedom coupled model.
- The parameters of the coupled SDOF model can be obtained by energy methods combined with the first model of vibration assumption.
- Two energy harvesting circuits are considered, namely (a) Harvesting circuit without an inductor, and (b) Harvesting circuit with an inductor
- The excitation is usually provided through a base excitation.
- The analysis can be carried out either in the time domain or in the frequency domain.
- A the equation of motion can be expressed in the original space or in the state space.

Energy harvesting circuits



(a) Harvesting circuit without an inductor



(b) Harvesting circuit with an inductor

Figure: Schematic diagrams of piezoelectric energy harvesters with two different harvesting circuits.

The equation of motion

- The coupled electromechanical behaviour of the energy harvester can be expressed by linear ordinary differential equations as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \theta v(t) = f_b(t) \tag{1}$$

$$C_p\dot{v}(t) + \frac{1}{R_l}v(t) + \theta\dot{x}(t) = 0 \tag{2}$$

- Here:
  $x(t)$: displacement of the mass
  m: equivalent mass of the harvester
  k: equivalent stiffness of the harvester
  c: damping of the harvester
  $f_b(t)$: base excitation force to the harvester
  θ: electromechanical coupling
  $v(t)$: voltage
  R_l: load resistance
  C_p: capacitance of the piezoelectric layer

The state-space equation

- The force due to base excitation is given by

$$f_b(t) = -m\ddot{x}_b(t) \tag{3}$$

- In the time domain, equations (1) and (2) can be expressed in the state-space form as

$$\frac{d\mathbf{z}_1(t)}{dt} = \mathbf{A}_1\mathbf{z}_1(t) + \mathbf{B}_1 f_b(t) \tag{4}$$

- The state-vector $\mathbf{z}$ and corresponding coefficient matrices are defined as

$$\mathbf{z}_1(t) = \begin{Bmatrix} x(t) \\ \dot{x}(t) \\ v(t) \end{Bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ -k/m & -c/m & \theta/m \\ 0 & -\theta/C_p & -1/(C_p R_l) \end{bmatrix}, \tag{5}$$

$$\text{and} \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix} \tag{6}$$

The state-space equation

- Equation (4) can be solved with suitable initial conditions and elements of the state-vector can be obtained.
- The solution will involve exponential of the matrix $\mathbf{A}_1$.
- The natural frequencies of the system can be obtained by solving the eigenvalue problem involving the matrix $\mathbf{A}_1$, that is, form the roots of the following equation

$$\det |\mathbf{A}_1 - \lambda \mathbf{I}_3| = 0 \tag{7}$$

- Here I_3 is a 3×3 identity matrix and the above equation will have 3 roots.
- As the matrix $\mathbf{A}_1$ is real and the system is stable, two roots will be in complex conjugate pairs and one root will be real and negative.
- The interest of this paper is to analyse the nature of the voltage $v(t)$ when the forcing function is a harmonic excitation.

Frequency domain: dimensional form

- Transforming equations (1) and (2) into the frequency domain we have

$$\begin{bmatrix} -m\omega^2 + c\mathrm{i}\omega + k & -\theta \\ \mathrm{i}\omega\theta & \mathrm{i}\omega C_p + \frac{1}{R_l} \end{bmatrix} \begin{Bmatrix} X(\omega) \\ V(\omega) \end{Bmatrix} = \begin{Bmatrix} F_b(\omega) \\ 0 \end{Bmatrix} \tag{8}$$

- Hence the frequency domain description of the displacement and the voltage can be obtained by inverting the coefficient matrix as

$$\begin{Bmatrix} X(\omega) \\ V(\omega) \end{Bmatrix} = \frac{1}{\Delta_1(\mathrm{i}\omega)} \begin{bmatrix} \mathrm{i}\omega C_p + \frac{1}{R_l} & \theta \\ -\mathrm{i}\omega\theta & -m\omega^2 + c\mathrm{i}\omega + k \end{bmatrix} \begin{Bmatrix} F_b \\ 0 \end{Bmatrix} \tag{9}$$

$$= \begin{Bmatrix} \left(\mathrm{i}\omega C_p + \frac{1}{R_l}\right) F_b/\Delta_1 \\ -\mathrm{i}\omega\theta F_b/\Delta_1 \end{Bmatrix} \tag{10}$$

- Here the determinant of the coefficient matrix is

$$\Delta_1(\mathrm{i}\omega) = mC_p(\mathrm{i}\omega)^3 + (m/R_l + cC_p)(\mathrm{i}\omega)^2 + \left(kC_p + \theta^2 + c/R_l\right)(\mathrm{i}\omega) + k/R_l \tag{11}$$

Frequency domain: non-dimensional form

- Transforming equations (1) and (2) into the frequency domain we obtain and dividing the first equation by m and the second equation by C_p we obtain

$$\left(-\omega^2 + 2\mathrm{i}\omega\zeta\omega_n + \omega_n^2\right) X(\omega) - \frac{\theta}{m} V(\omega) = F_b(\omega) \tag{12}$$

$$\mathrm{i}\omega \frac{\theta}{C_p} X(\omega) + \left(\mathrm{i}\omega + \frac{1}{C_p R_l}\right) V(\omega) = 0 \tag{13}$$

- Here $X(\omega)$, $V(\omega)$ and $F_b(\omega)$ are respectively the Fourier transforms of $x(t)$, $v(t)$ and $f_b(t)$.
- The natural frequency of the harvester, ω_n, and the damping factor, ζ, are defined as

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n} \tag{14}$$

Frequency domain: non-dimensional form

- Dividing the preceding equations by ω_n and writing in matrix form one has

$$\begin{bmatrix} \left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta & -\frac{\theta}{k} \\ \mathrm{i}\Omega\frac{\alpha\theta}{C_p} & \left(\mathrm{i}\Omega\alpha+1\right) \end{bmatrix} \begin{Bmatrix} X \\ V \end{Bmatrix} = \begin{Bmatrix} F_b \\ 0 \end{Bmatrix} \tag{15}$$

- Here the dimensionless frequency and dimensionless time constant are defined as

$$\Omega = \frac{\omega}{\omega_n} \quad \text{and} \quad \alpha = \omega_n C_p R_l \tag{16}$$

- The constant α is the time constant of the first order electrical system, non-dimensionalized using the natural frequency of the mechanical system.

Frequency domain: non-dimensional form

- Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} (\mathrm{i}\Omega\alpha + 1) & \frac{\theta}{k} \\ -\mathrm{i}\Omega\frac{\alpha\theta}{C_p} & \left(1-\Omega^2\right) + 2\mathrm{i}\Omega\zeta \end{bmatrix} \begin{Bmatrix} F_b \\ 0 \end{Bmatrix} = \begin{Bmatrix} (\mathrm{i}\Omega\alpha + 1)\, F_b/\Delta_1 \\ -\mathrm{i}\Omega\frac{\alpha\theta}{C_p} F_b/\Delta_1 \end{Bmatrix} \quad (17)$$

- The determinant of the coefficient matrix is

$$\Delta_1(\mathrm{i}\Omega) = (\mathrm{i}\Omega)^3\alpha + (2\,\zeta\,\alpha+1)\,(\mathrm{i}\Omega)^2 + \left(\alpha + \kappa^2\alpha + 2\,\zeta\right)(\mathrm{i}\Omega) + 1 \quad (18)$$

  This is a cubic equation in $\mathrm{i}\Omega$ leading to to three roots.
- The non-dimensional electromechanical coupling coefficient is

$$\kappa^2 = \frac{\theta^2}{kC_p} \quad (19)$$

The equation of motion

- The coupled electromechanical behaviour of the energy harvester can be expressed by linear ordinary differential equations as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) - \theta v(t) = f_b(t) \tag{20}$$

$$C_p\ddot{v}(t) + \frac{1}{R_l}\dot{v}(t) + \frac{1}{L}v(t) + \theta\ddot{x}(t) = 0 \tag{21}$$

- Here L is the inductance of the circuit. Note that the mechanical equation is the same as given in equation (1).
- Unlike the previous case, these equations represent two coupled second-order equations and opposed one coupled second-order and one first-order equations.

The state-space equation

- Equations (20) and (21) can be expressed in the state-space form as

$$\frac{d\mathbf{z}_2(t)}{dt} = \mathbf{A}_2\mathbf{z}_2(t) + \mathbf{B}_2 f_b(t) \tag{22}$$

- Here the state-vector $\mathbf{z}$ and corresponding coefficient matrices are defined as

$$\mathbf{z}_2(t) = \begin{Bmatrix} x(t) \\ \dot{x}(t) \\ v(t) \\ \dot{v}(t) \end{Bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 1/m \\ 0 \\ -\theta/mC_p \end{bmatrix} \quad \text{and} \tag{23}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m & -c/m & \theta/m & 0 \\ 0 & 0 & 0 & 1 \\ \theta k/mC_p & \theta c/mC_p & \theta^2/mC_p - 1/LC_p & -1/RC_p \end{bmatrix} \tag{24}$$

Equation (22) can be solved in a similar way as Equation (4).

The state-space equation

- Equation (22) can be solved with suitable initial conditions and elements of the state-vector can be obtained.
- The solution will involve exponential of the matrix $\mathbf{A}_2$.
- The natural frequencies of the system can be obtained by solving the eigenvalue problem involving the matrix $\mathbf{A}_2$, that is, form the roots of the following equation

$$\det\left|\mathbf{A}_2 - \lambda \mathbf{I}_4\right| = 0 \tag{25}$$

- Here I_4 is a 4×4 identity matrix and the above equation will have 4 roots.
- As the matrix $\mathbf{A}_2$ is real and the system is stable, two roots will be in complex conjugate pairs and the other two roots will be real and negative.

Frequency domain: dimensional form

- Transforming equation (21) into the frequency domain one obtains

$$-\omega^2\theta X(\omega) + \left(-\omega^2 C_p + \mathrm{i}\omega\frac{1}{R_l} + \frac{1}{L}\right) V(\omega) = 0 \tag{26}$$

- Similar to equation (15), this equation can be written in matrix form with the equation of motion of the mechanical system as

$$\begin{bmatrix} -m\omega^2 + c\mathrm{i}\omega + k & -\theta \\ -\omega^2\theta & -\omega^2 C_p + \mathrm{i}\omega\frac{1}{R_l} + \frac{1}{L} \end{bmatrix} \begin{Bmatrix} X(\omega) \\ V(\omega) \end{Bmatrix} = \begin{Bmatrix} F_b(\omega) \\ 0 \end{Bmatrix} \tag{27}$$

Frequency domain: dimensional form

- Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X(\omega) \\ V(\omega) \end{Bmatrix} = \frac{1}{\Delta_2} \begin{bmatrix} -\omega^2 C_p + \mathrm{i}\omega \frac{1}{R_l} + \frac{1}{L} & \theta \\ \omega^2 \theta & -m\omega + c\mathrm{i}\omega + k \end{bmatrix} \begin{Bmatrix} F_b \\ 0 \end{Bmatrix} = \begin{Bmatrix} \left(-\omega^2 C_p + \mathrm{i}\omega \frac{1}{R_l} + \frac{1}{L}\right) F_b/\Delta_2 \\ \omega^2 \theta F_b/\Delta_2 \end{Bmatrix} \quad (28)$$

- Here the determinant of the coefficient matrix is a fourth-order polynomial in $(\mathrm{i}\omega)$ and is given by

$$\Delta_2(\mathrm{i}\omega) = mC_p(\mathrm{i}\omega)^4 + \frac{(cC_pR_lL + mL)}{R_lL}(\mathrm{i}\omega)^3 + \frac{(mR_l + cL + \theta^2 R_lL + kC_pR_lL)}{R_lL}(\mathrm{i}\omega)^2 + \frac{(cR_l + kL)}{R_lL}(\mathrm{i}\omega) + \frac{k}{L} \quad (29)$$

Frequency domain: non-dimensional form

- Transforming equation (21) into the frequency domain and dividing by $C_p\omega_n^2$ one has

$$-\Omega^2\frac{\theta}{C_p}X+\left(-\Omega^2+\mathrm{i}\Omega\frac{1}{\alpha}+\frac{1}{\beta}\right)V=0 \tag{30}$$

- The second dimensionless constant is defined as

$$\beta=\omega_n^2 L C_p \tag{31}$$

  This is the ratio of the mechanical to electrical natural frequencies.
- Similar to Equation (15), this equation can be written in matrix form with the equation of motion of the mechanical system (12) as

$$\begin{bmatrix}\left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta & -\frac{\theta}{k}\\ -\Omega^2\frac{\alpha\beta\theta}{C_p} & \alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta\end{bmatrix}\begin{Bmatrix}X\\ V\end{Bmatrix}=\begin{Bmatrix}F_b\\ 0\end{Bmatrix} \tag{32}$$

Frequency domain: non-dimensional form

- Inverting the coefficient matrix, the displacement and voltage in the frequency domain can be obtained as

$$\begin{Bmatrix} X \\ V \end{Bmatrix} = \frac{1}{\Delta_2} \begin{bmatrix} \alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta & \frac{\theta}{k} \\ \Omega^2\frac{\alpha\beta\theta}{C_p} & \left(1-\Omega^2\right)+2\mathrm{i}\Omega\zeta \end{bmatrix} \begin{Bmatrix} F_b \\ 0 \end{Bmatrix} = \begin{Bmatrix} \left(\alpha\left(1-\beta\Omega^2\right)+\mathrm{i}\Omega\beta\right) F_b/\Delta_2 \\ \Omega^2\frac{\alpha\beta\theta}{C_p} F_b/\Delta_2 \end{Bmatrix} \quad (33)$$

- The determinant of the coefficient matrix is

$$\Delta_2(\mathrm{i}\Omega) = (\mathrm{i}\Omega)^4\beta\,\alpha + (2\,\zeta\,\beta\,\alpha+\beta)(\mathrm{i}\Omega)^3 + \left(\beta\,\alpha+\alpha+2\,\zeta\,\beta+\kappa^2\beta\,\alpha\right)(\mathrm{i}\Omega)^2 + (\beta+2\,\zeta\,\alpha)(\mathrm{i}\Omega)+\alpha \quad (34)$$

This is a quartic equation in $\mathrm{i}\Omega$ leading to to four roots.

Summary

- The single-degree-of-freedom coupled model can effectively represent a piezoelectric Euler-Bernoulli beam with a tip mass.
- Dynamic analysis of the coupled SDOF is discussed in the time domain and in the frequency domain.
- Two circuit configurations have been introduced, namely, (a) Energy harvesters without an inductor and (b) Energy harvesters with an inductor
- The first case leads to a state-space system of dimension three and the model has three roots for it eigenvalues
- The second case leads to a state-space system of dimension four and the model has four roots for it eigenvalues
- Explicit expressions of displacement and voltage response in the frequency domain for both the cases have been derived in closed-form.