

# Part 1: Reduced-order modelling for vibration energy harvesting

Professor Sondipon Adhikari  
FRAeS

Chair of Aerospace Engineering, College of Engineering, Swansea University, Swansea UK  
Email: [S.Adhikari@swansea.ac.uk](mailto:S.Adhikari@swansea.ac.uk), Twitter: [@ProfAdhikari](https://twitter.com/ProfAdhikari)  
Web: <http://engweb.swan.ac.uk/~adhikaris>  
Google Scholar: <http://scholar.google.co.uk/citations?user=tKM35S0AAAAJ>

October 30, 2017



Swansea University



# Swansea University





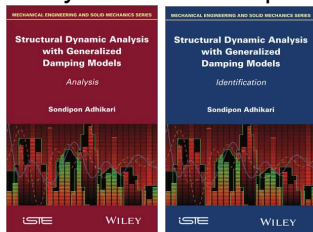
## Brief Biography of Professor Adhikari

- MSc (Engineering), [Indian Institute of Science, Bangalore](#) [08/1995–09/1997]
- PhD in Engineering, [University of Cambridge](#), Trinity College, UK [10/1997–07/2001]
- **Past Positions:** Assistant Prof, Department of Aerospace Engineering, [University of Bristol](#) [01/2003–03/2007]
- **Current Positions:** Full Prof (Aerospace Engineering), [Swansea University](#) [04/2007–]
- **Awards:** Nehru Cambridge Scholarship; Overseas Research Student Award,; Rouse-Ball Travelling Scholarship from Trinity College, Cambridge; Philip Leverhulme Prize; Associate Fellow of the American Institute of Aeronautics and Astronautics; Wolfson Research Merit Award from the Royal Society, London; EPSRC Ideas factory winner.
- **Professional activities:** Editorial board member of 20 journals, 270 journal papers,  $h$ -index=50

# My research interests

- *Development* of fundamental computational methods for structural dynamics and uncertainty quantification

## A. Dynamics of complex systems



## B. Inverse problems for linear and non-linear dynamics

## C. Uncertainty quantification in computational mechanics

- *Applications* of computational mechanics to emerging multidisciplinary research areas

## D. Vibration energy harvesting / dynamics of wind turbines

## E. Computational nanomechanics

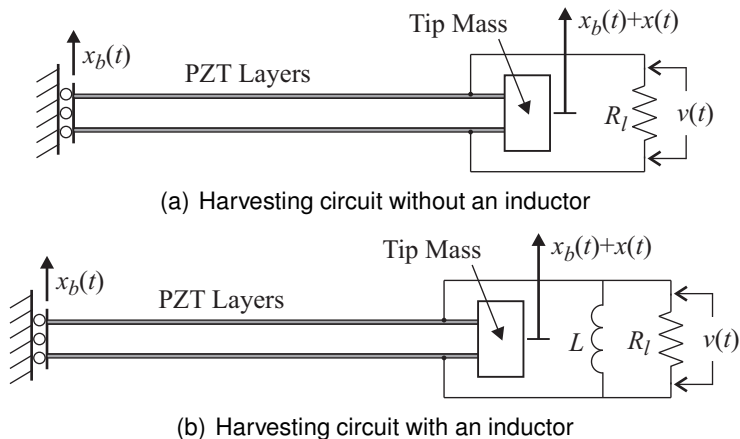
## Outline of this talk

- 1 **Introduction**
- 2 **Euler-Bernoulli theory for vibrating cantilevers**
- 3 **Dynamics of beams with a tip mass**
- 4 **Finite element approach**
- 5 **Reduced order modelling**
  - Derivation of a single-degree-of-freedom model
  - Dynamics of SDOF systems
- 6 **Electromechanical coupling**
- 7 **Equivalent single-degree-of-freedom coupled model**
- 8 **Summary**

## Energy Harvesting systems

- Vibration energy harvesting for micro-scale systems is normally realised using a **cantilever beam** with a piezoelectric patch.
- The excitation is usually provided through a **base excitation**.
- A **proof mass** is often added at the end of the cantilever to adjust the frequency so that the first natural frequency of the overall beam is close to the primary excitation frequency.
- The analysis is therefore focused on a **piezoelectric beam with a tip mass** driven by a harmonic base excitation.
- A **single mode approximation** is often employed to simplify the analysis. This is generally suitable as most of the energy of the system is confined around the first mode of vibration.
- This in turn can be achieved by an **equivalent single degree of freedom model** approximation.

# Euler-Bernoulli beam theory



**Figure:** Schematic diagrams of piezoelectric energy harvesters with two different harvesting circuits.



## Euler-Bernoulli beam theory

- Due to the small thickness to length ratio, **Euler-Bernoulli beam theory** can be used to model bending vibration of energy harvesting cantilevers.

The **equation of motion** of a damped cantilever modelled (see for example [1]) using Euler-Bernoulli beam theory can be expressed as

$$EI \frac{\partial^4 U(x, t)}{\partial x^4} + \hat{c}_1 \frac{\partial^5 U(x, t)}{\partial x^4 \partial t} + \rho A \frac{\partial^2 U(x, t)}{\partial t^2} + \hat{c}_2 \frac{\partial U(x, t)}{\partial t} = F(x, t) \quad (1)$$

- In the above equation  $x$  is the coordinate along the length of the beam,  $t$  is the time,  $E$  is the Young's modulus,  $I$  is the second-moment of the cross-section,  $A$  is the cross-section area,  $\rho$  is the density of the material,  $F(x, t)$  is the applied spatial dynamic forcing and  $U(x, t)$  is the transverse displacement.

## Euler-Bernoulli beam theory

- The length of the beam is assumed to be  $L$ .
- This equation is **fourth-order partial differential equation** with constant coefficients.
- The constant  $\hat{c}_1$  is the strain-rate-dependent viscous damping coefficient,  $\hat{c}_2$  is the velocity-dependent viscous damping coefficient.
- The **strain-rate-dependent** viscous damping can be used to model inherent damping property of the material of the cantilever beam.
- The **velocity-dependent** viscous damping can be used to model damping due to external factors such as a cantilever immersed in an fluidic environment.

## Euler-Bernoulli beam theory

- Schematic diagrams of piezoelectric energy harvesters with **two different harvesting circuits** are shown in 1.
- We first consider the free vibration of the beam without the tip mass.
- The **boundary conditions** for cantilevered Euler-Bernoulli dictates that the deflections and rotation at the supported end is zero and the bending moment and shear force at the free end are zero.
- At  $x = 0$

$$U(x, t) = 0, \quad \frac{\partial U(x, t)}{\partial x} = 0 \quad (2)$$

- At  $x = L$

$$\frac{\partial^2 U(x, t)}{\partial x^2} = 0, \quad \frac{\partial^3 U(x, t)}{\partial x^3} = 0 \quad (3)$$

## Undamped free vibration solution

- Dynamics of the beam is characterised by the **undamped free vibration solution**. This in turn can be expressed by undamped **natural frequency** and vibration **mode shapes**.
- We assume the **separation of variables** as

$$U(x, t) = u(t)\phi(x) \quad (4)$$

The spatial function is further expressed as

$$\phi(x) = e^{\lambda x} \quad (5)$$

- Substituting this in the original partial differential equation (1) and applying the boundary conditions [2], one obtains the **undamped natural frequencies** (rad/s) of a cantilever beam as

$$\omega_j = \lambda_j^2 \sqrt{\frac{EI}{\rho AL^4}}, \quad j = 1, 2, 3, \dots \quad (6)$$

## Undamped free vibration solution

- The constants  $\lambda_j$  needs to be obtained by solving the following transcendental equation

$$\cos \lambda \cosh \lambda + 1 = 0 \quad (7)$$

- Solving this equation [3], the values of  $\lambda_j$  can be obtained as 1.8751, 4.69409, 7.8539 and 10.99557. For larger values of  $j$ , in general we have  $\lambda_j = (2j - 1)/2\pi$ .
- The **vibration mode shape** corresponding to the  $j$ -th natural frequency can be expressed as

$$\phi_j(\xi) = (\cosh \lambda_j \xi - \cos \lambda_j \xi) - \left( \frac{\sinh \lambda_j - \sin \lambda_j}{\cosh \lambda_j + \cos \lambda_j} \right) (\sinh \lambda_j \xi - \sin \lambda_j \xi) \quad (8)$$

where  $\xi = \frac{x}{L}$  is the normalised coordinate along the length of the cantilever.

## Cantilever beam with a tip mass

- The effect of the tip mass is incorporated using the boundary condition at the free edge.
- The new boundary condition is therefore:

At  $x = L$

$$\frac{\partial^2 U(x, t)}{\partial x^2} = 0, \quad EI \frac{\partial^3 U(x, t)}{\partial x^3} - M \frac{\partial^2 U(x, t)}{\partial t^2} = 0 \quad (9)$$

- It can be shown that (see for example [4]) the resonance frequencies are still obtained from Eq. (6) but  $\lambda_j$  should be obtained by solving

$$(\cos \lambda \sinh \lambda - \sin \lambda \cosh \lambda) \Delta M \lambda + (\cos \lambda \cosh \lambda + 1) = 0 \quad (10)$$

- Here

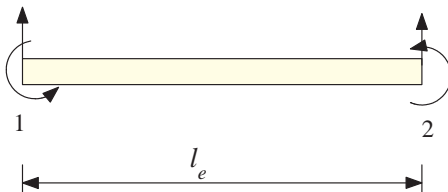
$$\Delta M = \frac{M}{\rho AL} \quad (11)$$

is the ratio of the added mass and the mass of the cantilever.

- If the added mass is zero, then one can see that Eq. (11) reduces to Eq. (7). For this general case, the eigenvalues  $\lambda_j$  as well as the mode shapes  $\phi_j(\xi)$  become a function of  $\Delta M$ .
- Unlike the classical mass-free case, **closed-form expressions are not available**. However, very accurate approximation can be developed for this case [4].
- An **energy based method** is developed to consider the first mode of vibration, leading to a reduced order model.

## Finite element method

- We consider an element of length  $\ell_e$  with bending stiffness  $EI$  and mass per unit length  $m$ .



**Figure:** A nonlocal element for the bending vibration of a beam. It has two nodes and four degrees of freedom. The displacement field within the element is expressed by cubic shape functions.

- This element has four degrees of freedom and there are four shape functions.



## Element stiffness matrix

- The shape function matrix for the bending deformation [5] can be given by

$$\mathbf{N}(x) = [N_1(x), N_2(x), N_3(x), N_4(x)]^T \quad (12)$$

where

$$\begin{aligned} N_1(x) &= 1 - 3\frac{x^2}{\ell_e^2} + 2\frac{x^3}{\ell_e^3}, & N_2(x) &= x - 2\frac{x^2}{\ell_e} + \frac{x^3}{\ell_e^2}, \\ N_3(x) &= 3\frac{x^2}{\ell_e^2} - 2\frac{x^3}{\ell_e^3}, & N_4(x) &= -\frac{x^2}{\ell_e} + \frac{x^3}{\ell_e^2} \end{aligned} \quad (13)$$

- Using this, the stiffness matrix can be obtained using the conventional variational formulation [6] as

$$\mathbf{K}_e = EI \int_0^{\ell_e} \frac{d^2 \mathbf{N}(x)}{dx^2} \frac{d^2 \mathbf{N}^T(x)}{dx^2} dx = \frac{EI}{\ell_e^3} \begin{bmatrix} 12 & 6\ell_e & -12 & 6\ell_e \\ 6\ell_e & 4\ell_e^2 & -6\ell_e & 2\ell_e^2 \\ -12 & -6\ell_e & 12 & -6\ell_e \\ 6\ell_e & 2\ell_e^2 & -6\ell_e & 4\ell_e^2 \end{bmatrix}$$

## Element mass matrix

- The mass matrix for the nonlocal element can be obtained as

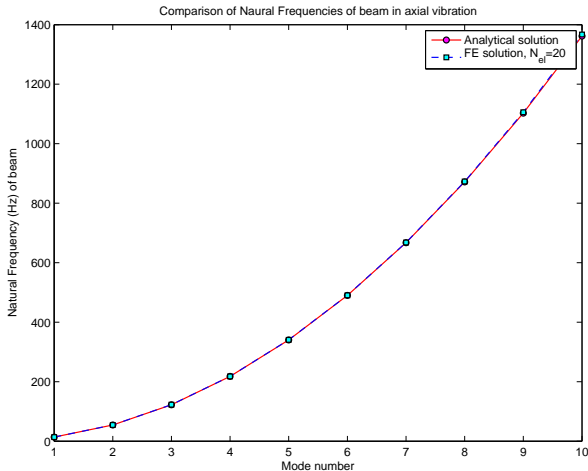
$$\begin{aligned}\mathbf{M}_e &= m \int_0^{\ell_e} \mathbf{N}(x) \mathbf{N}^T(x) dx \\ &= \frac{m\ell_e}{420} \begin{bmatrix} 156 & 22\ell_e & 54 & -13\ell_e \\ 22\ell_e & 4\ell_e^2 & 13\ell_e & -3\ell_e^2 \\ 54 & 13\ell_e & 156 & -22\ell_e \\ -13\ell_e & -3\ell_e^2 & -22\ell_e & 4\ell_e^2 \end{bmatrix} \end{aligned} \quad (15)$$

- The element damping matrix can be obtained using the linear combination of the mass and stiffness matrices as

$$\mathbf{C}_e = \alpha \mathbf{M}_e + \beta \mathbf{K}_e \quad (16)$$

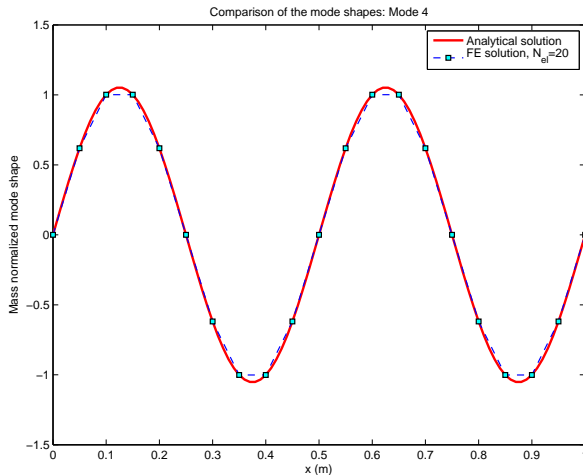
- The element matrices should be assembled to form the global matrices

## Comparison of natural frequencies



**Figure:** Comparison of natural frequencies between the exact analytical result and the finite element result.

# Comparison of mode shapes



**Figure:** Comparison of a mode shape between the exact analytical result and the finite element result.

## Reduced order model for a cantilever with a tip mass

- The equation of motion of the beam in (1) is a partial differential equation. This equation represents infinite number of degrees of freedom.
- The mathematical theory of linear partial differential equations is very well developed and the nature of solutions of the bending vibration is well understood.
- Considering the steady-state harmonic motion with frequency  $\omega$  we have

$$U(x, t) = u(x) \exp [i\omega t] \quad (17)$$

$$\text{and } F(x, t) = f(x) \exp [i\omega t] \quad (18)$$

where  $i = \sqrt{-1}$ .

## Reduced order model for a cantilever with a tip mass

- Substituting this in the beam equation (1) we have

$$EI \frac{d^4 u(x)}{dx^4} + i\omega \hat{c}_1 \frac{d^4 u(x)}{dx^4} - \rho A \omega^2 u(x) + i\omega \hat{c}_2 u(x) = f(x) \quad (19)$$

- Following the damping convention in dynamic analysis as in [7], we consider stiffness and mass proportional damping.
- Therefore, we express the damping constants as

$$\hat{c}_1 = \alpha(EI) \quad \text{and} \quad \hat{c}_2 = \beta(\rho A) \quad (20)$$

where  $\alpha$  and  $\beta$  are stiffness and mass proportional damping factors.

## Reduced order model for a cantilever with a tip mass

- Substituting these, from Eq. (19) we have

$$EI \frac{d^4 u(x)}{dx^4} + i\omega \underbrace{\left( \alpha EI \frac{d^4 u(x)}{dx^4} + \beta \rho A u(x) \right)}_{\text{damping}} - \rho A \omega^2 u(x) = f(x) \quad (21)$$

- The first part of the damping expression is proportional to the stiffness term while the second part of the damping expression is proportional the mass term.
- The general solution of Eq. (21) can be expressed as a linear superposition of all the vibration mode shapes (see for example [7]).

## Reduced order model for a cantilever with a tip mass

- Piezoelectric vibration energy harvesters are often designed to operate within a frequency range which is close to first few natural frequencies only.
- Therefore, without any loss of accuracy, simplified lumped parameter models can be used to corresponding correct resonant behaviour.
- This can be achieved using energy methods or more generally using Galerkin approach.
- Galerkin approach can be employed in the time domain or in the frequency domain. We adopt a time domain approach. The necessary changes to apply this in the frequency domain is straightforward for linear problems and therefore not elaborated here.



## Reduced order model for a cantilever with a tip mass

- Assuming a unimodal solution, the dynamic response of the beam can be expressed as

$$U(x, t) = u_j(t)\phi_j(x), \quad j = 1, 2, 3, \dots \quad (22)$$

- Substituting this assumed motion into the equation of motion (1), multiplying by  $\phi_j(x)$  and integrating by parts over the length one has

$$\begin{aligned} Elu_j(t) \int_0^L \phi_j''^2(x) dx + \alpha El\dot{u}_j(t) \int_0^L \phi_j''^2(x) dx \\ + \beta \rho A \dot{u}_j(t) \int_0^L \phi_j^2(x) dx + \rho A \ddot{u}_j(t) \int_0^L \phi_j^2(x) dx \\ = \int_0^L F(x, t) \phi_j(x) dx \end{aligned} \quad (23)$$

## Reduced order model for a cantilever with a tip mass

- Using the equivalent mass, damping and stiffness, this equation can be rewritten as

$$m_{eqj} \ddot{u}_j(t) + c_{eqj} \dot{u}_j(t) + k_{eqj} u_j(t) = f_j(t) \quad (24)$$

where the equivalent mass and stiffness terms are given by

$$m_{eqj} = \rho A \int_0^L \phi_j^2(x) dx = \rho A L \underbrace{\int_0^1 \phi_j^2(\xi) d\xi}_{l_{1j}} \quad (25)$$

$$k_{eqj} = EI \int_0^L \phi_j''^2(x) dx = \frac{EI}{L^3} \underbrace{\int_0^1 \phi_j''^2(\xi) d\xi}_{l_{2j}} \quad (26)$$

## Reduced order model for a cantilever with a tip mass

- The equivalent damping and the equivalent forcing are expressed as

$$c_{eqj} = \alpha k_{eqj} + \beta m_{eqj} \quad (27)$$

$$f_j(t) = \int_0^L F(x, t) \phi_j(x) dx \quad (28)$$

- Using the expression of the mode-shape in Eq. (8), the integrals  $l_{1j}$  and  $l_{2j}$  can be evaluated in closed-form for any general mode as

$$\begin{aligned} l_{1j} = & (-\cos \lambda_j \sinh \lambda_j - \cos^2 \lambda_j \cosh \lambda_j \sinh \lambda_j \\ & + \lambda_j \cos^2 \lambda_j - \cosh \lambda_j \sin \lambda_j - \cos \lambda_j \cosh^2 \lambda_j \sin \lambda_j \\ & + 2 \cos \lambda_j \cosh \lambda_j \lambda_j + \lambda_j \cosh^2 \lambda_j) / (D \lambda_j) \end{aligned} \quad (29)$$

## Reduced order model for a cantilever with a tip mass



$$I_{2j} = \lambda_j^3 (3 \cos \lambda_j \sinh \lambda_j + 3 \cos^2 \lambda_j \cosh \lambda_j \sinh \lambda_j + \lambda_j \cos^2 \lambda_j + 3 \cosh \lambda_j \sin \lambda_j + \lambda_j \cosh^2 \lambda_j + 3 \cos \lambda_j \cosh^2 \lambda_j \sin \lambda_j + 2 \cos \lambda_j \cosh \lambda_j \lambda_j) / D \quad (30)$$

- The denominator  $D$  is given by

$$D = \cosh^2 \lambda_j + 2 \cos \lambda_j \cosh \lambda_j + \cos^2 \lambda_j \quad (31)$$

- For the first mode of vibration ( $j = 1$ ), substituting  $\lambda_1 = 1.8751$ , it can be shown that

$$I_{11} = 1$$

and

$$I_{12} = 12.3624.$$

## Reduced order model for a cantilever with a tip mass

- If there is a point mass of  $M$  at the tip of the cantilever, then the effective mass becomes

$$m_{eq_j} = \rho AL l_j + \underbrace{M \phi_j^2(1)}_{l_{3_j}} = \rho AL \left( l_j + \Delta M l_{3_j} \right) \quad (32)$$

- Using the expression of the mode-shape we have

$$l_{3_j} = \frac{4 \sinh^2 \lambda_j \sin^2 \lambda_j}{D} \quad (33)$$

- For the first mode of vibration it can be shown that  $l_{3_1} = 4$ .
- The equivalent single degree of freedom model given by Eq. (24) will be used in the rest of the talk.
- However, the expression derived here are general and can be used if higher modes of vibration were to be employed in energy harvesting.

## Dynamics of damped SDOF system

- The equation of motion of the equivalent single degree of freedom cantilever harvester is expressed as

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = f(t) \quad (34)$$

where

$$c = \alpha k + \beta m \quad (35)$$

- We call the oscillator given by Eq. (34) as the reference oscillator. Diving by  $m$ , the equation of motion can be expressed as

$$\ddot{u}(t) + 2\zeta\omega_n\dot{u}(t) + \omega_n^2 u(t) = \frac{f(t)}{m} \quad (36)$$

## Dynamics of damped SDOF system

- Here the undamped natural frequency ( $\omega$ ) and the damping factor ( $\zeta$ ) are expressed as

$$\omega_n = \sqrt{\frac{k}{m}} \quad (37)$$

$$\text{and } \frac{c}{m} = 2\zeta\omega_n \quad \text{or} \quad \zeta = \frac{c}{2\sqrt{km}} \quad (38)$$

- In view of the expression of  $c$  in (35), the damping factor can also be expressed in terms of the stiffness and mass proportional damping constants as

$$\zeta = \frac{1}{2} \left( \alpha\omega_n + \frac{\beta}{\omega_n} \right) \quad (39)$$

## Dynamics of damped SDOF system

- Taking the Laplace transform of Eq. (36) we have

$$s^2 U(s) + s2\zeta\omega_n U(s) + \omega_n^2 U(s) = \frac{F(s)}{m} \quad (40)$$

where  $U(s)$  and  $F(s)$  are the Laplace transforms of  $u(t)$  and  $f(t)$  respectively.

- Solving the equation associated with coefficient of  $U(s)$  in Eq. (36) without the forcing term, the complex natural frequencies of the system are given by

$$s_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} = -\zeta\omega_n \pm i\omega_d \quad (41)$$

- Here the imaginary number  $i = \sqrt{-1}$  and the damped natural frequency is expressed as

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \quad (42)$$



## Dynamics of damped SDOF system

- For a damped oscillator, at resonance, the frequency of oscillation is given by  $\omega_d < \omega_n$ . Therefore, for positive damping, the resonance frequency of a damped system is always lower than the corresponding underlying undamped system.
- The Quality factor (Q-factor) of an oscillator is the ratio between the energy stored and energy lost during one cycle when the oscillator vibrates at the resonance frequency.
- It can be shown that the Q-factor

$$Q = \frac{m\omega_d}{c} = \frac{\sqrt{1 - \zeta^2}}{2\zeta} \quad (43)$$

- Alternatively, the damping factor can be related to the Q-factor as

$$\zeta = \frac{1}{\sqrt{1 + 4Q^2}} \quad (44)$$

We will use both factors as appropriate.

## Dynamics of damped SDOF system

- Assuming the amplitude of the harmonic excitation as  $F$ , from the Laplace transform expression in Eq. (40), the response in the frequency domain can be expressed by substituting  $s = i\omega$  as

$$\left(-\omega^2 + i\omega 2\zeta\omega_n + \omega_n^2\right) U(i\omega) = \frac{F}{m} \quad (45)$$

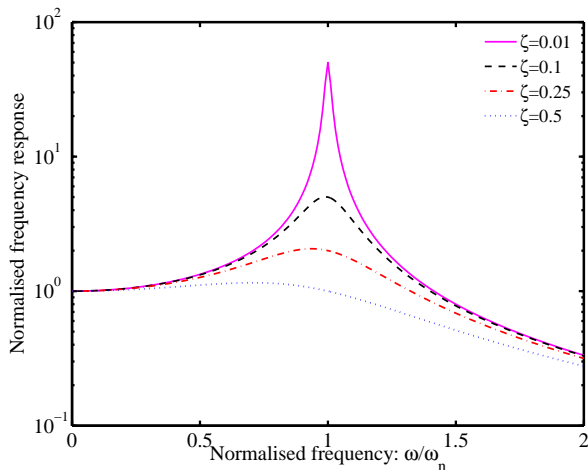
- Dividing this by  $\omega^2$ , the frequency response function of the mass-absorbed oscillator can be expressed as

$$U(i\Omega) = \frac{U_{st}}{-\Omega^2 + 2i\Omega\zeta + 1} \quad (46)$$

where the normalised frequency and the static response are given by

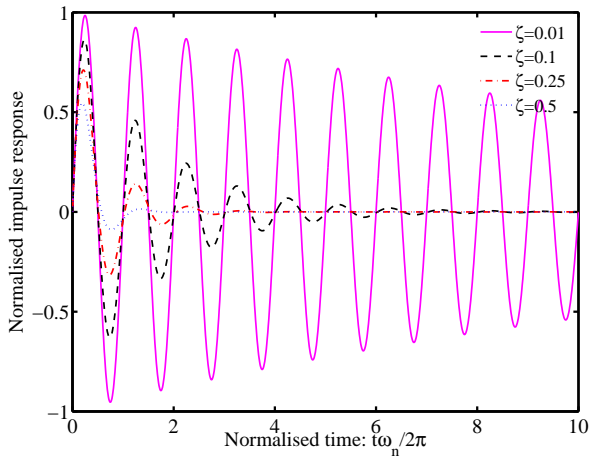
$$\Omega = \frac{\omega}{\omega_n} \quad \text{and} \quad U_{st} = \frac{F}{k} \quad (47)$$

# Frequency response function



**Figure:** Normalised amplitude of the frequency response function for various values of the damping factor  $\zeta$

## Impulse response function



**Figure:** Normalised impulse response function for various values of the damping factor  $\zeta$

## Coupled electromechanical model

- So far we have not used any piezoelectric effect. Here we will take this into account.
- Suppose that piezoelectric layers added to a beam in either a unimorph or a bimorph configuration. Then the moment about the beam neutral axis produced by a voltage  $V$  across the piezoelectric layers may be written as

$$M(x, t) = \gamma_c V(t) \quad (48)$$

- The constant  $\gamma_c$  depends on the geometry, configuration and piezoelectric device and  $V(t)$  is the time-dependent voltage.
- For a bimorph with piezoelectric layers in the 31 configuration, with thickness  $h_c$ , width  $b_c$  and connected in parallel

$$\gamma_c = Ed_{31}b_c(h + h_c) \quad (49)$$

where  $h$  is the thickness of the beam and  $d_{31}$  is the piezoelectric constant.

## Coupled electromechanical model

- For a unimorph, the constant is

$$\gamma_c = Ed_{31}b_c \left( h + \frac{h_c}{2} - \bar{z} \right) \quad (50)$$

where  $\bar{z}$  is the effective neutral axis

- These expressions assume a monolithic piezoceramic actuator perfectly bonded to the beam.
- The work done by the piezoelectric patches in moving or extracting the electrical charge is

$$W = \int_0^{L_c} M(x, t) \kappa(x) dx \quad (51)$$

where  $L_c$  is the active length of the piezoelectric material, which is assumed to be attached at the clamped end of the beam.

## Coupled electromechanical model

- The quantity  $\kappa(x)$  is the curvature of the beam and this is approximately expressed by the second-derivative of the displacement
- Using the approximation for  $\kappa$  we have

$$W = \theta V \quad (52)$$

where the coupling coefficient

$$\theta = \gamma_c \int_0^{L_c} \frac{\partial^2 \phi(x)}{\partial x^2} dx = \gamma_c \phi'(L_c) \quad (53)$$

- Using the first mode shape, for  $L_c = L$ , this can be evaluated as

$$\phi'(1) = 2L \frac{\lambda (\cos(\lambda) \sinh(\lambda) + \cosh(\lambda) \sin(\lambda))}{\cosh(\lambda) + \cos(\lambda)} \quad (54)$$

## Coupled electromechanical oscillator

- For the harvesting circuit **without** an inductor, the coupled electromechanical behaviour can be expressed by the linear ordinary differential equations

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) - \theta v(t) = f_b(t) \quad (55)$$

$$C_p \dot{v}(t) + \frac{1}{R_l} v(t) + \theta \dot{x}(t) = 0 \quad (56)$$

- For the harvesting circuit **with** an inductor, the electrical equation becomes

$$\theta \ddot{u}(t) + C_p \ddot{v}(t) + \frac{1}{R_l} \dot{v}(t) + \frac{1}{L_l} v(t) = 0 \quad (57)$$

- Here

$$m = \rho AL (l_1 + \Delta M l_3), k = \frac{EI}{L^3} l_2, \theta = \gamma_c L l_4 \quad (58)$$



## Coupled electromechanical oscillator

- The integrals

$$I_1 = \int_0^1 \phi^2(\xi) d\xi, I_2 = \int_0^1 \phi''^2(\xi) d\xi, I_3 = \phi^2(1), I_4 = \phi'(\xi_c) \quad (59)$$

- Equation (55) is simply Newton's equation of motion for a single degree of freedom system, where  $t$  is the time,  $x(t)$  is the displacement of the mass,  $m$ ,  $c$  and  $k$  are respectively the modal mass, modal damping and modal stiffness of the harvester and  $x_b(t)$  is the base excitation.
- The electrical load resistance is  $R_l$ ,  $L_l$  is the inductance,  $\theta$  is the electromechanical coupling, and the mechanical force is modelled as proportional to the voltage across the piezoceramic,  $v(t)$ .
- Equation (56) is obtained from the electrical circuit, where the voltage across the load resistance arises from the mechanical strain through the electromechanical coupling,  $\theta$ , and the capacitance of the piezoceramic,  $C_p$ .

## Coupled electromechanical oscillator

- The first cantilever mode is given by

$$\phi(\xi) = (\cosh \lambda_1 \xi - \cos \lambda_1 \xi) - \sigma_1 (\sinh \lambda_1 \xi - \sin \lambda_1 \xi) \quad (60)$$

where

$$\sigma_1 = \frac{\sinh \lambda_1 - \sin \lambda_1}{\cosh \lambda_1 + \cos \lambda_1} \quad (61)$$

- Using the  $\lambda_1 = 1.8751$  for the first mode, the quantities can be evaluated numerically as

$$\sigma_1 = 0.7341, l_1 = 1, l_2 = 12.3624, l_3 = 4, l_4 = 2.7530 \quad (\xi_c = 1) \quad (62)$$

- Here, the force due to base excitation is given by

$$f_b(t) = -m\ddot{x}_b(t) \quad (63)$$

- The book by Erturk and Inman [8] gives further details on this model.

## Summary

- Euler-Bernoulli theory for vibrating cantilever beams has been introduced.
- Dynamic analysis of beams with a tip mass is discussed in details.
- Finite element approach for Euler-Bernoulli beams has been briefly covered.
- Reduced order modelling by expressing the solution in terms of the undamped eigenmodes of the beam has been explained. It was shown that by retaining the first mode, the reduced equation can be expressed by a single-degree-of-freedom (SDOF) model.
- The idea of electromechanical coupling has been explained and mathematical derivation in terms of the assumed mode has been developed.
- It was shown that the dynamics of a piezoelectric beam with a tip mass can be expressed in terms of coupled a SDOF electromechanical model.
- All necessary coefficients are derived in closed-form.

## Further reading

- [1] H. T. Banks, D. J. Inman, On damping mechanisms in beams, Transactions of ASME, Journal of Applied Mechanics 58 (1991) 716–723.
- [2] D. J. Inman, Engineering Vibration, Prentice Hall PTR, NJ, USA, 2003.
- [3] R. D. Blevins, Formulas for Natural Frequency and Mode Shape, Krieger Publishing Company, Malabar, FL, USA, 1984.
- [4] S. Adhikari, R. Chowdhury, The calibration of carbon nanotube based bio-nano sensors, Journal of Applied Physics 107 (12) (2010) 124322:1–8.
- [5] M. Petyt, Introduction to Finite Element Vibration Analysis, Cambridge University Press, Cambridge, UK, 1998.
- [6] D. Dawe, Matrix and Finite Element Displacement Analysis of Structures, Oxford University Press, Oxford, UK, 1984.
- [7] L. Meirovitch, Principles and Techniques of Vibrations, Prentice-Hall International, Inc., New Jersey, 1997.
- [8] A. Erturk, D. J. Inman, Piezoelectric Energy Harvesting: Modelling and Application, Wiley-Blackwell, Sussex, UK, 2011.