Dr Francesco Mezzadri
School of Mathematics

Some Applications of Random Matrix Theory

Uncertainty in Structural Dynamics, 20 April 2006

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Outline

- What is Random Matrix Theory (RMT) about?
- Applications of RMT to:
  - Number Theory
  - Quantum Chaos
  - Quantum Information

  } University of Bristol specialities

  - Wireless Telecommunications
  - Much more not discussed in this talk (Probability, Combinatorics, Structural dynamics,…)

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What is RMT about?

Consider a random $N \times N$ matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ m_{N1} & m_{N2} & \cdots & m_{NN} \end{pmatrix}$$

The questions that we would like to answer are:

1. How are the eigenvalues correlated?
2. Can we compute integrals of the type

$$\int_{\mathcal{M}} f(M)dM \quad ?$$

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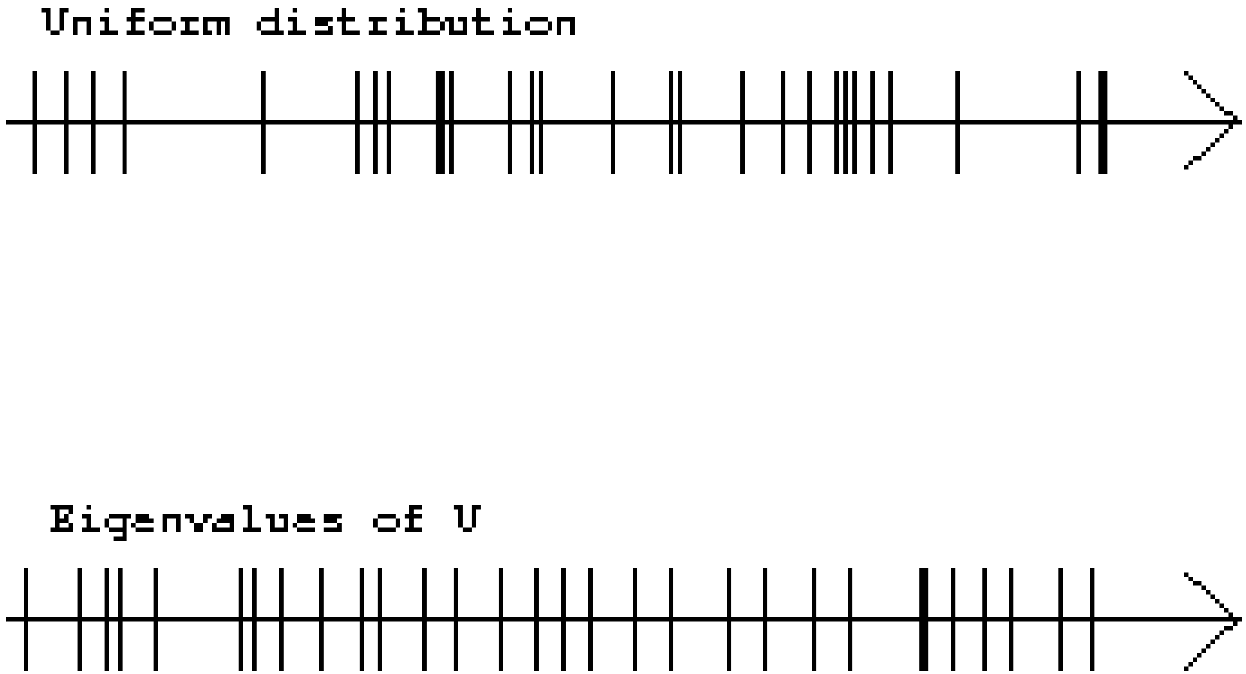

Comparison between a sequence of random numbers uniformly distributed and the spectrum of a random Hermitian matrix

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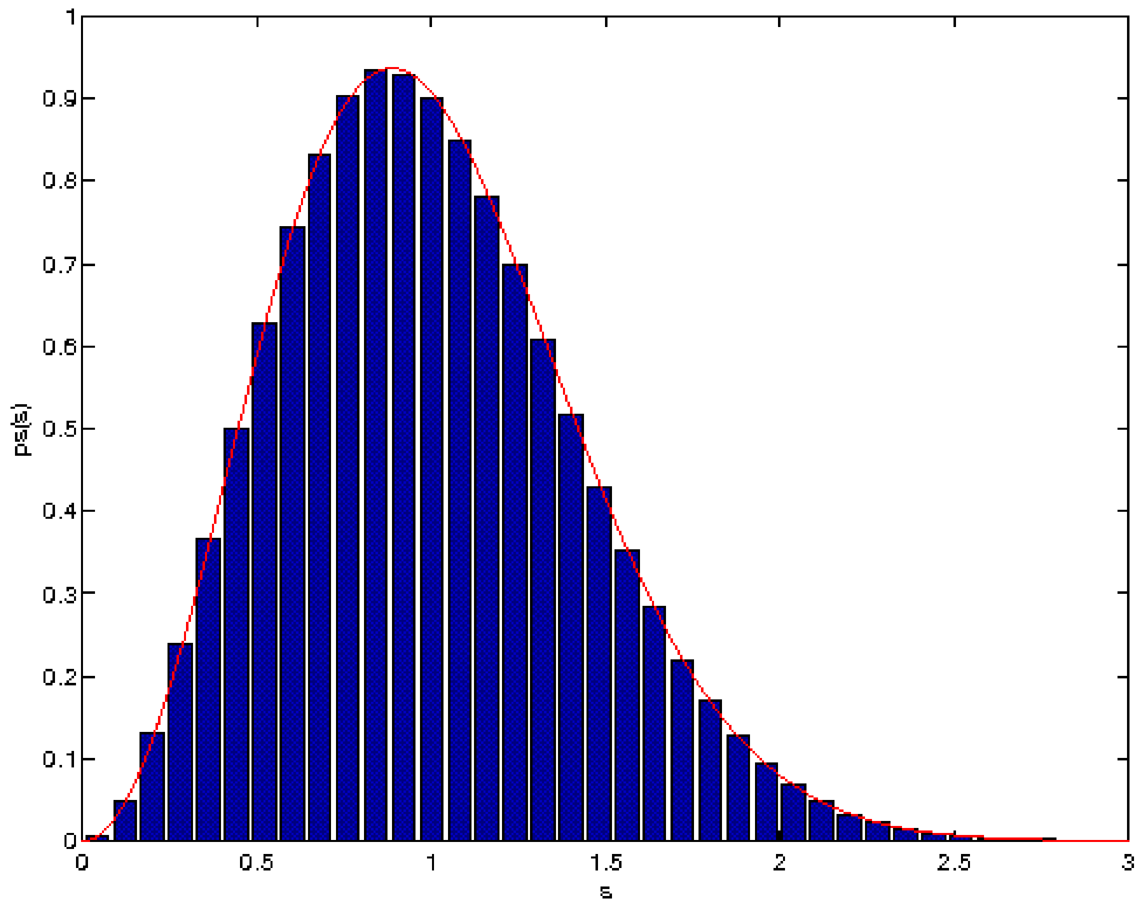

Spacing distribution of 50×50 matrices in the CUE ensemble

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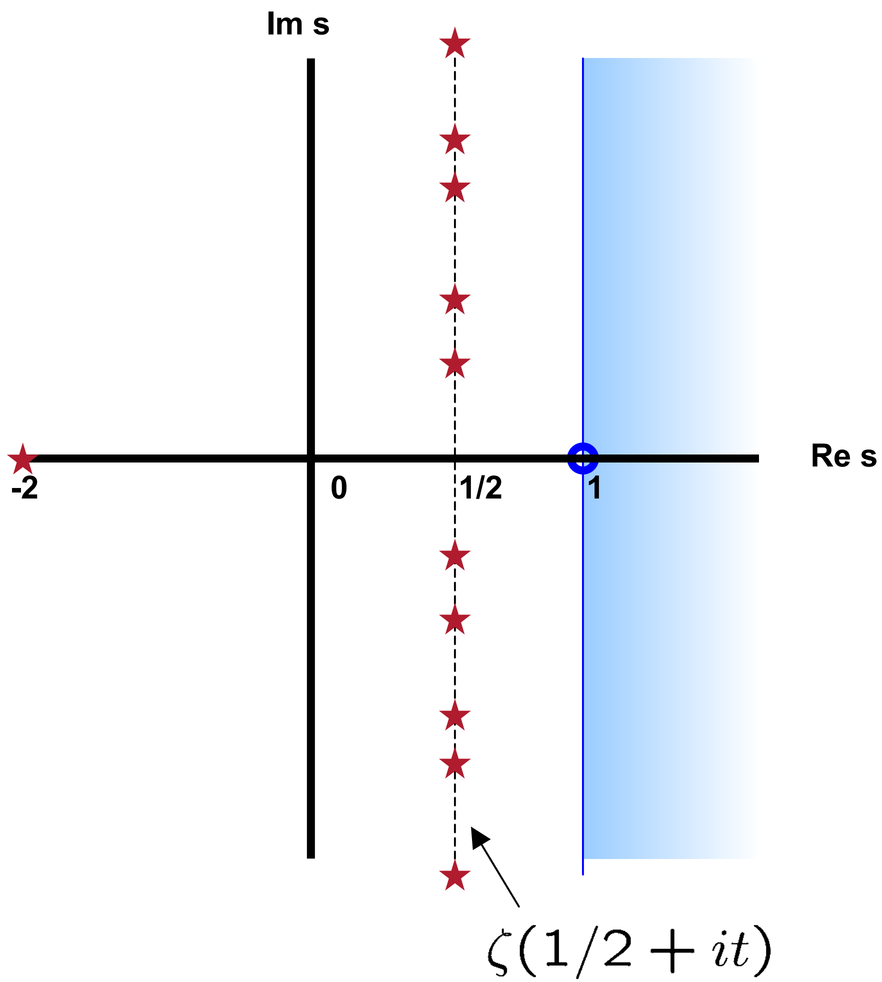
RMT and the Riemann zeta function

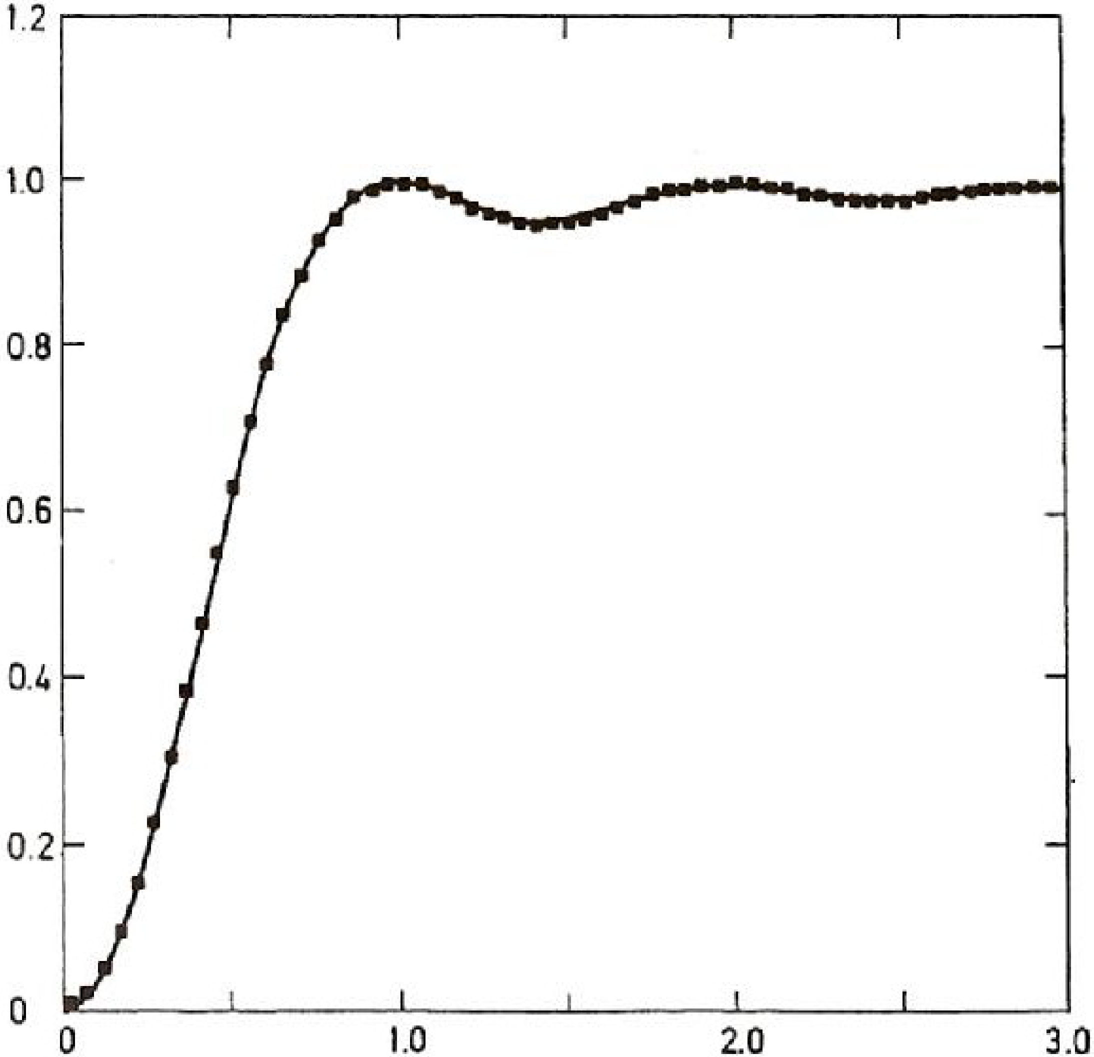


For Re$s > 1$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$



Picture by A. Odlyzko

79 million zeros around the 10^{20} th zero

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

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Moments of the Riemann zeta function

Conrey and Ghosh suggested that as $T \to \infty$:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt \sim a_\lambda \frac{g_\lambda}{\Gamma(1+\lambda^2)} \log^{\lambda^2} T$$

with $a_\lambda = \prod_p \left[(1-1/p)^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\lambda)}{\Gamma(\lambda) m!}\right)^2 p^{-m}\right]$, a product over prime numbers

$$\frac{g_0}{\Gamma(1)} = 1$$

$$\frac{g_1}{\Gamma(1+1)} = 1 \quad \text{theorem – Hardy\&Littlewood 1918}$$

$$\frac{g_2}{\Gamma(1+2^2)} = \frac{1}{12} \quad \text{theorem – Ingham 1926}$$

$$\frac{g_3}{\Gamma(1+3^2)} = \frac{42}{9!} \quad \text{conjecture – Conrey\&Ghosh 1992}$$

$$\frac{g_4}{\Gamma(1+4^2)} = \frac{24024}{16!} \quad \text{conjecture – Conrey\&Gonek 2001}$$

Conjecture (Keating and Snaith 2000):

$$\lim_{T\to\infty}(\log T)^{-\lambda^2}\frac{1}{T}\int_0^T|\zeta(1/2+it)|^{2\lambda}dt$$

$$= a_\lambda \lim_{N\to\infty} N^{-\lambda^2}\int_{\mathsf{U}(N)}|\Lambda_U(e^{i\theta})|^{2\lambda}dU_{\mathsf{Haar}}$$

where $\Lambda_U(s) = \prod_{n=1}^{N}\left(1-se^{-i\theta_n}\right) = \det\left(I-sU^\dagger\right)$

It turns out that

$$\int_{\mathsf{U}(N)}\left|\Lambda_U(e^{i\theta})\right|^{2\lambda}dU_{\mathsf{Haar}} = \prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+2\lambda)}{(\Gamma(j+\lambda))^2}$$

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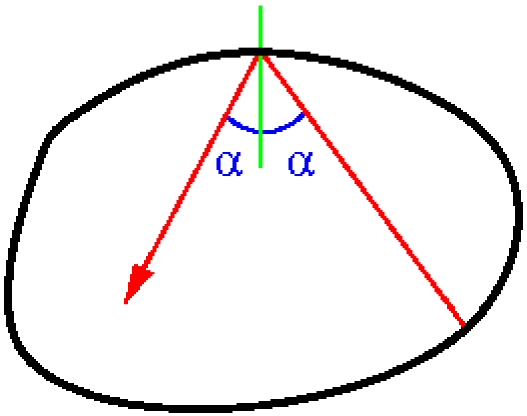
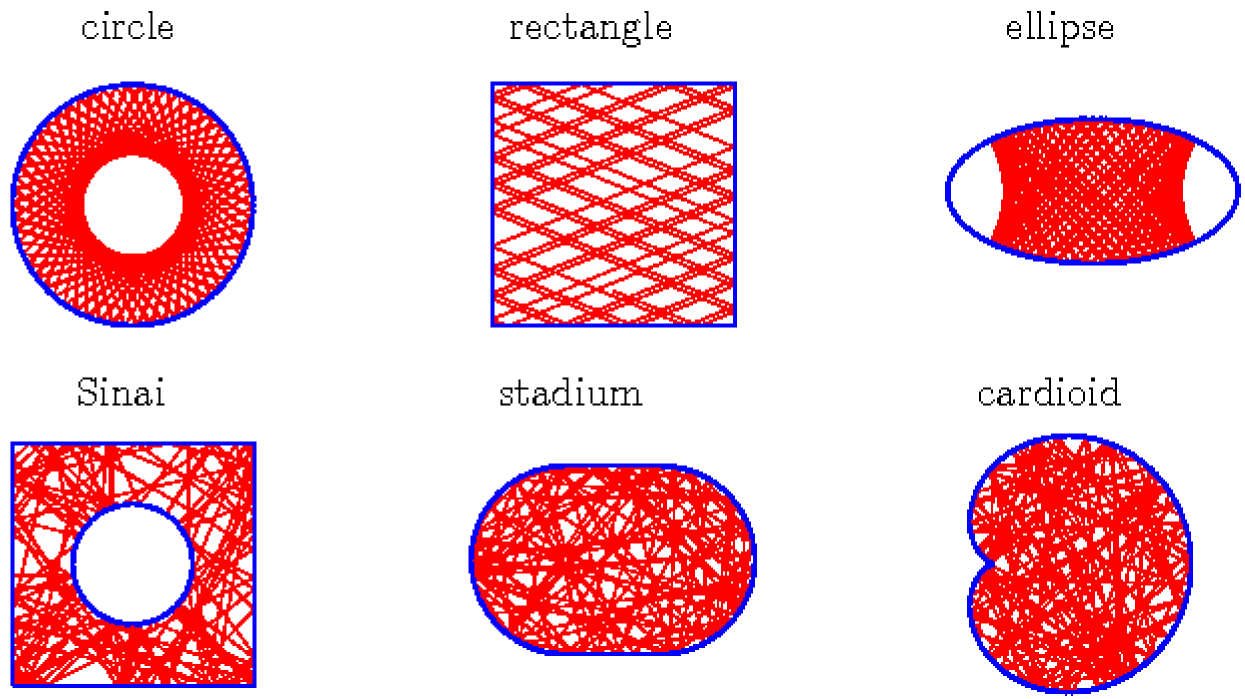
Quantum Chaos and RMT

Free particle moving in a compact domain:



- The particle moves at constant speed in straight line
- Angle of incidence = angle of reflection



Pictures courtesy of Arnd Bäcker

The quantum mechanics of billiards is described by Helmholtz's equation:

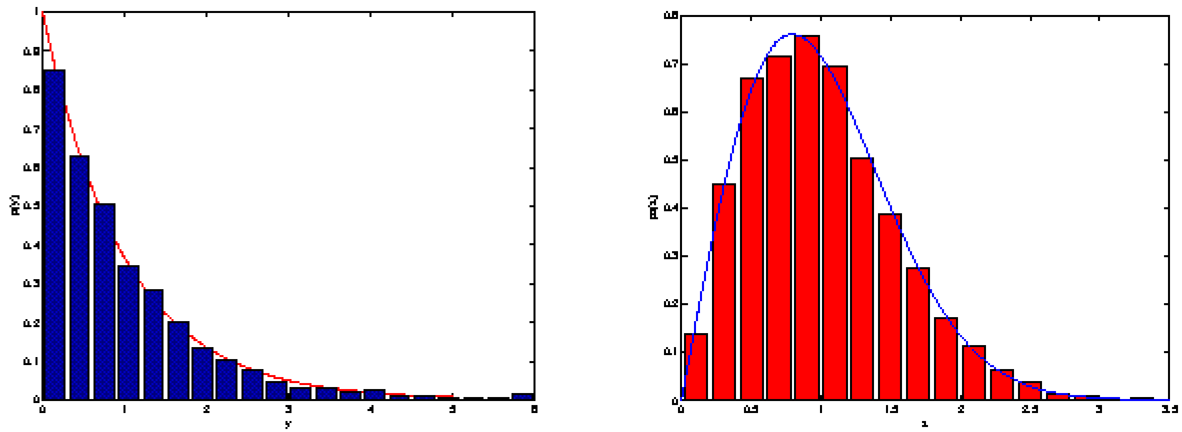
$$\triangle\psi_n(\mathbf{r}) + k_n^2\psi_n(\mathbf{r}) = 0.$$

We then order the eigenvalues

$$k_0 \leq k_1 \leq k_2 \leq k_3 \leq \ldots \leq k_n \leq \ldots$$

One of the most important question in quantum chaos is:

How does the 'chaotic' or 'regular' behaviour of the classical dynamics affects the distribution of the eigen-values of Helmholtz's equation in the limit as $n \to \infty$



(a) Regular dynamics (b) Chaotic dynamics

Figure 1: Comparison of the histograms of the spacings between consecutive eigenvalues of a dynamical system whose classical dynamics is regular and one whose dynamics is chaotic.

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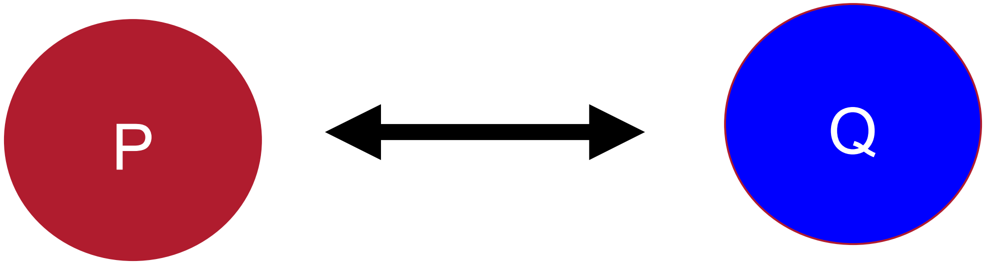
Applications: mesoscopic systems

These are systems that contain a large number of atoms but are small compared to macroscopic objects. They live on the boundary between the macroscopic and microscopic world. Mesoscopic physics address fundamental physical problems which occur when macroscopic objects are miniaturized. Example: quantum dots.

Entanglement and RMT

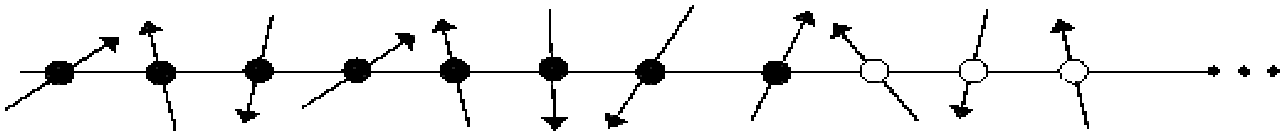
Entanglement is the ability that quantum systems have of exhibit correlations that cannot be accounted for classically

All the physical information of a quantum system is contained in a vector $\psi \in \mathcal{H}$



Bipartite system: $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_Q$

Entanglement in quantum spin chains



The entropy of entanglement is

$$E(\Psi_{\mathrm{g}}) = \lim_{\epsilon \to 0+} \lim_{\delta \to 0+} \frac{1}{2\pi \mathrm{i}} \oint_{c(\epsilon,\delta)} \mathrm{e}(1+\epsilon, \lambda) \frac{\mathrm{d} \ln D_N[g](\lambda)}{\mathrm{d}\lambda} \mathrm{d}\lambda$$

where

$$\mathrm{e}(x,\nu) = -\frac{x+\nu}{2} \log_2\left(\frac{x+\nu}{2}\right) - \frac{x-\nu}{2} \log_2\left(\frac{x-\nu}{2}\right)$$

and

$$D_N[g](\lambda) = \int_{\mathrm{G}(N)} g(U) dU_{\mathrm{Haar}}$$

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It turns out that (Keating and Mezzadri 2004, 2005; Keating, Mezzadri and Novaes 2006)

$$E(\Psi_g) \sim -R\frac{2^{w(\sigma_1,\sigma_2)}}{6}\log_2 N$$

- We can compute entanglement for many families spin chains in many different instances.
- One-to-one correspondence between symmetries of the spin chains and matrix ensembles.
- Computation of higher order terms.
- Connection with conformal field theory.

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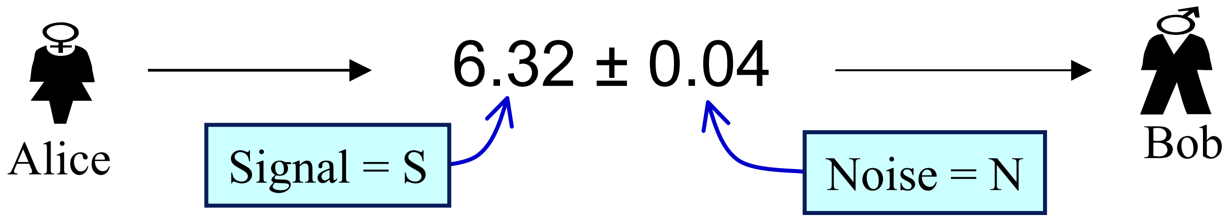
RMT and Multiantenna communications

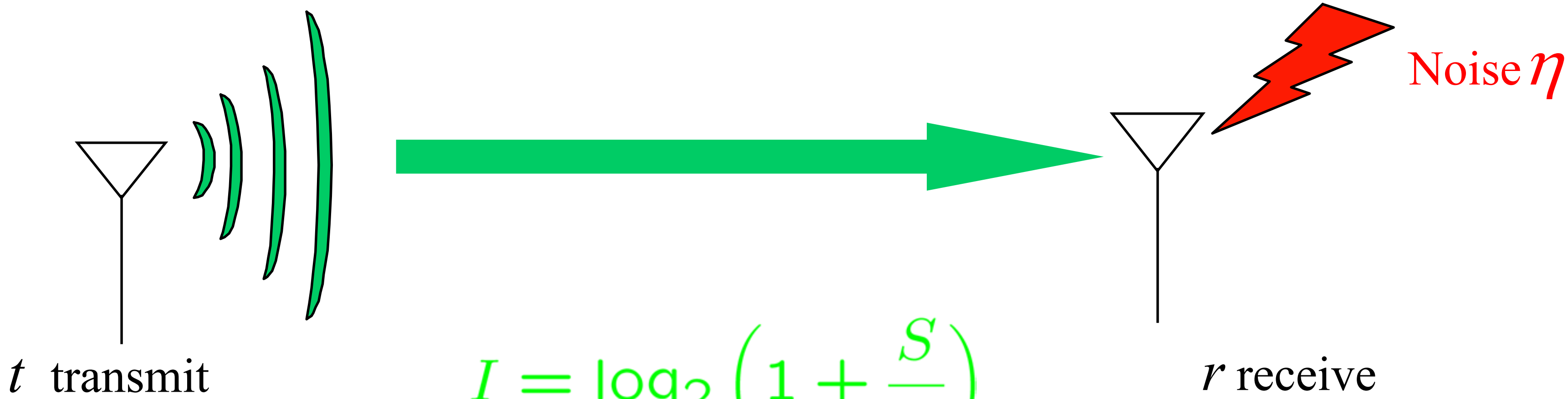


Theorem (Shannon 1948). *If the signal is chosen from a complex normal distribution and the noise is complex normal then*

$$I = \log_2\left(1 + \frac{S}{N}\right).$$

(signal power) — S

(noise power) — N



$$I = \log_2\left(1 + \frac{S}{N}\right)$$

We have $r = Gt + \eta$

Transfer (Green's) function

$$I = \log_2\left(1 + |G|^2 P_t / N\right)$$

Transmitted power

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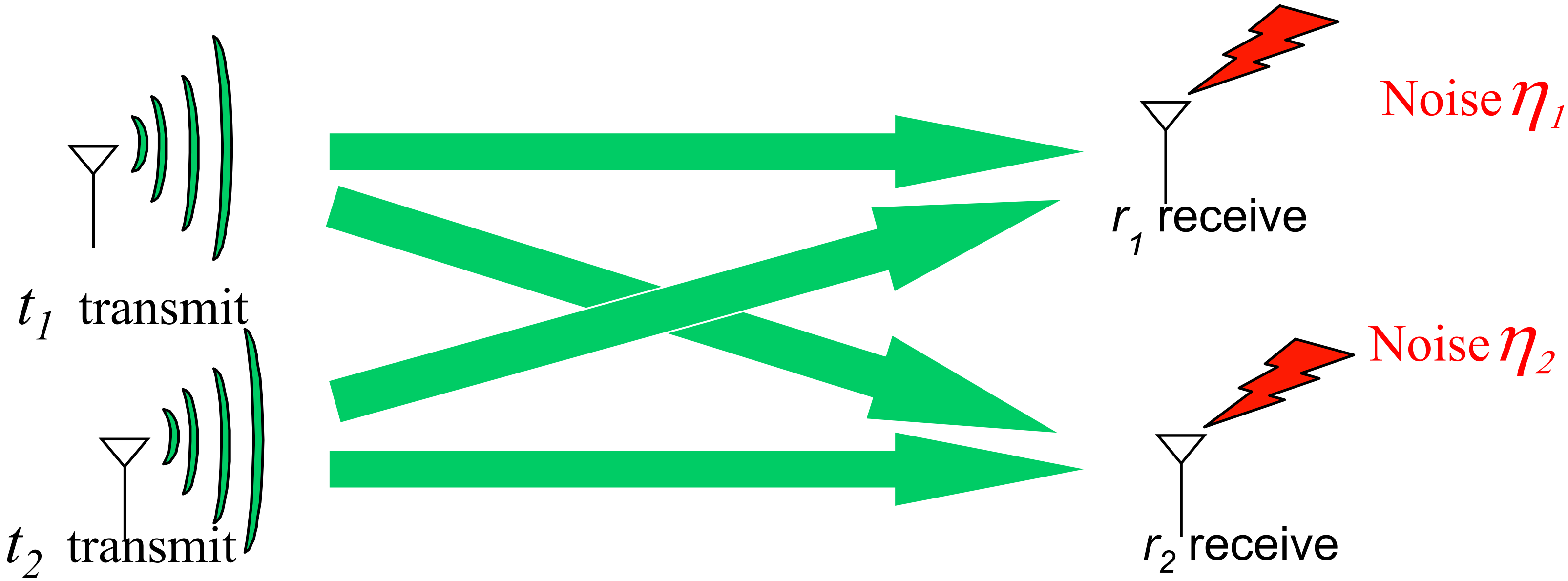

$$t_j = \sum_k G_{jk} r_k + \eta_j$$

Transfer (Green's) matrix

$$I = \log_2 \det\left(1 + GG^{\dagger} P_t / N\right).$$

- Because the environment keeps changing *G* is a random matrix;
- Ensembles of matrices of the form $GG^{\dagger}$ are called *Wishart ensembles.*
- Computing

$$I = \log_2 \det\left(1 + GG^{\dagger} P_t/N\right).$$

reduces to compute ensemble averages

$$\int_{\mathcal{M}} g(G)dG$$

Thanks to Steve Simon for introducing me to applications of RMT in telecommunications and for some of the pretty pictures in these slides

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