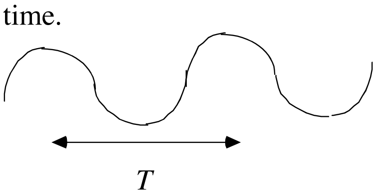
## §6 a.c. (Alternating Current) Circuits

Most signals of interest in electronics are periodic : they repeat regularly as a function of time.




$T$, the time after a waveform repeats itself, is called the period.

The frequency, measured in Hertz (Hz) or cycles per second ,is the inverse of the period.

$$f = \frac{1}{T}$$

For example mains voltage repeats 50 times per second, so $f$ = 50 Hz. The period (for repetition) is then

$$T = \frac{1}{f} = 20\ ms$$

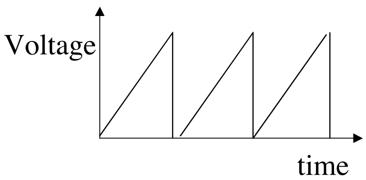
At a frequency of 1 GHz ($10^9$ Hz) we have



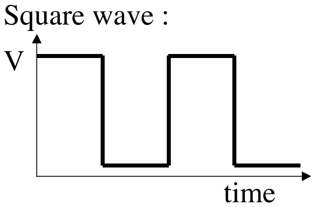


$$T = \frac{1}{f} = 1\ ns$$

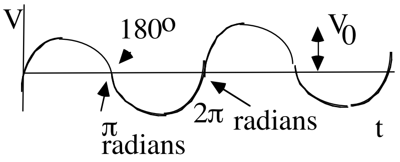
Common waveforms include : Sawtooth (used in oscilloscopes & TV raster scans).




Square wave :




However the most important periodic waveform is the sine/cosine. Any periodic waveform can be built up from / synthesised from a sum of sines and cosines (Fourier's theorem).




We have to translate between the angular measure we normally associate with trigonometric functions, e.g. π radians or 180° degrees, and the time, which we would measure when displaying the waveform on a scope screen for example.

Commonly have a sin wave :
$V = V_0 \sin\omega t$

To find the relationship between ω, the frequency f and the period T think...

Half way through the period $\omega t = \pi$, or $t = \dfrac{\omega}{\pi}$
all the way through the period

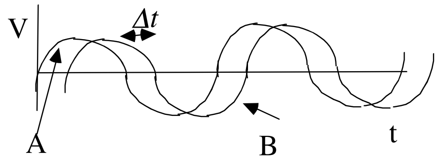
$$\omega t = \omega T = 2\pi, \quad T = \frac{2\pi}{\omega}, \quad \omega = \frac{2\pi}{T} = 2\pi f$$





*ω*, is known as the angular frequency and is measured in radians per second.

As soon as a circuit has some reactance (arising from capacitance or inductance) as well as resistance we will have sine waves which are displaced in time relative to one another (weird things capacitors & inductors).




Waveform B lags waveform A by a time difference *Δt*. This is usually expressed as an angle, the phase difference.

$$\phi = \frac{\Delta t}{T} \times 2\pi \text{ radians}, \quad \phi = \frac{\Delta t}{T} \times 360 \text{ degrees}$$

waveform A: $V = V_0 \sin(\omega t)$
waveform B: $V = V_0 \sin(\omega t - \phi)$





The effect of the majority of electrical circuits on an input signal is to

a) Leave the frequency unaltered
b) Change the magnitude $V_0 \Rightarrow V_1$
c) Change the phase ø = phase difference

To describe circuit components we will want a set of mathematical rules from which we can calculate the amplitude and the phase change.

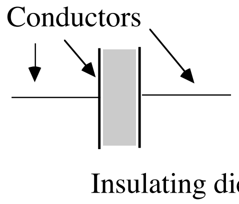




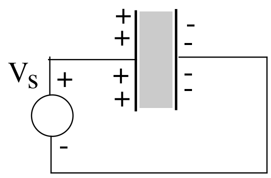
## §7 Capacitors, and complex numbers…

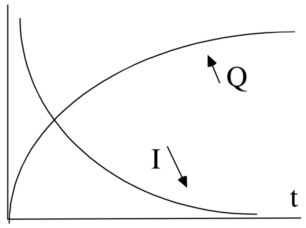
First we look at the most important components which give a phase change: the capacitor and the inductor. These components store electrical energy in the electromagnetic field. Capacitors store energy in the electric field and inductors in the magnetic field.

The basic structure of a capacitor is




If we connect the capacitor to a voltage source:






After a transient (current for a short time), no current flows as a result of a dc voltage. The capacitor acts as an open circuit to constant voltages across it.

The capacitance is a measure of the charge storage capacity.

The charge stored is the Capacitance in Farads times the Voltage
$Q = CV$
Now the current is the rate of change of charge so

$$I = \frac{dQ}{dt} = \frac{dCV}{dt} = C\frac{dV}{dt}$$





This means that the voltage cannot change discontinuously across a capacitor, since that would need an infinite current. Capacitors smooth out fast voltage changes. This may or may not be desirable, It may be a good thing if you want to smooth a power supply, but a terrible thing if you want to change voltage quickly in a high speed computer circuit! In engineering design sometimes may want to maximise, and sometimes minimise capacitance.

Ohm's law $V = IR$, has proved to be very useful. Can it somehow be extended to handle capacitors (and inductors) as well?

We need something which not only handles changes in the size of a signal (resistors may reduce it or amplifiers increase it) but also deals with any change in the phase of a signal as well.

It turns out there is a simple (but complex) elegant (though at first sight ugly) and powerful (though problematical to get started) solution to this. ☺





We generalise the notion of resistance to include the idea of **reactance** (phase shifting) using complex numbers. We call this combined thing (resistance plus reactance) an **impedance**.

To handle reactive elements such as capacitors and inductors we define a complex impedance Z which includes the resistance R and the reactance X. Impedance is a generalised complex resistance which can handle those pesky phase shifts.

The impedance $Z = a + jb\ \Omega$, where $a$ and $b$ are the real and complex parts of the impedance and $j^2 = -1$.
Note that if $j^2 = -1$, then $j = -1/j$

The impedance of a capacitor of capacitance C is given by.

$$Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$





In electronics $j$ is simply a quarter turn in phase (a 90 degree phase shift). There is nothing imaginary about this phase shift!

There are well-known electronic components which do this if we want to (inductors and capacitors), and also other components which do it even if we don't want them to (amplifiers, transistors, bits of wire almost anything at high frequency).

The reason we use complex numbers is that it is always easier to use the maths which best mimics what actually happens, and that means complex numbers are GREAT. Complex numbers make electronics easier because:

- They better represent what's happening.
- Allow straightforward fix to Ohm's law
- Easier to add impedances
- Easier to do products and quotients
- Easier to find square roots and squares
- Much easier to do calculus (and dynamics).

This is because the form of the maths mirrors the form of the underlying dynamics. That's what works best.

Right, we have had a version of Ohm's law before $V=IR$. It's not wrong, it just is not quite everything yet. Here it is, here is the new improved Ohm's law:

$V = I Z$

The difference is that, here, $V$, $I$ and $Z$ are complex.





The impedance of a capacitor is proportional, not to capacitance, but to 1/capacitance: the inverse capacitance.

$$Z = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

Now impedances are not only measured in Ohms, but combine in series and parallel in the same way as resistors.

Series $\quad Z_{total} = Z_1 + Z_2$

Parallel $\quad \dfrac{1}{Z_{total}} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$

Since capacitance is an inverse impedance (times some funny constants) **capacitors behave in the opposite way to resistors**:

*Capacitances in parallel add*
*Capacitances in series add inversely*

*Capacitors in parallel* $\quad C_{total} = C_1 + C_2$
This makes sense: two capacitors in parallel should behave in the same way as one capacitor with a bigger plate area.





Capacitors in series $\quad \dfrac{1}{C_{total}} = \dfrac{1}{C_1} + \dfrac{1}{C_2}$

The inductance of an inductor of $L$ henries is $Z = j\omega L$. Inductance is proportional to impedance (times some funny constants). This means that inductors add in the same way as resistances. Capacitors are the odd ones out.

Series $\quad L_{total} = L_1 + L_2$

Parallel $\quad \dfrac{1}{L_{total}} = \dfrac{1}{L_1} + \dfrac{1}{L_2}$ (like resistors).

Inductors consist of a magnetic field generator (usually a coil of wire), where magnetic field is generated when a current passes through it.

Any attempt to change the current produces a voltage at the terminals which is proportional to the rate of change of current through it.

That is the voltage is proportional to the rate of change of current. Mathematically the (extra) voltage is proportional to $dI/dt$.





The fact that a differential relationship exists means that if the current is a sin (or a cos) the voltage will be a cos (or a -sin). Just as was the case for the capacitor there will therefore be a phase difference between them.

The voltage has polarity which attempts to maintain the current at whatever value it is presently (so that the magnetic field will not change).

$$V = L\frac{dI}{dt}$$

the constant of proportionality $L$ is the inductance (of the coil) measured in Henries (H). Typical values are in the range mH to H.

Suppose $I = I_0 \sin(\omega t)$

$$V = L\frac{dI}{dt} = L\frac{d}{dt}\left(I_0 \sin\omega t\right)$$

$$= L\omega I_0 \cos(\omega t) = L\omega I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

For a resistor we would have $V = IR$ and by analogy, for an inductor we will have $V = Z_L I$ where $Z_L \approx L\omega$ except for the little problem of the phase change…
For the inductor the voltage is 90 degrees ahead of the current (current lags the voltage). Because $V = L\, dI/dt$ the current cannot change discontinuously (otherwise the voltage would go to infinity). The inductor acts a short circuit to a constant current, but as an open circuit to a very high frequency signal.
Contrast this with the capacitor :
For the capacitor the current is 90 degrees ahead of the voltage. Because $I = C\, dV/dt$ the voltage cannot change discontinuously. A capacitor is a short circuit to a high frequency signal and an open circuit to a d.c. source.

Impedances:

| Component | Impedance | Type |
|---|---|---|
| Resistor | $R$ | Real |
| Inductor | $j\omega L$ | Imaginary |
| Capacitor | $\frac{-j}{\omega C}$ | Imaginary |





Impedances are combined just like resistors:

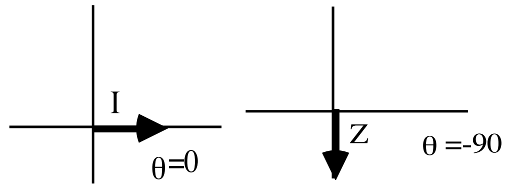
Series $Z_{total} = Z_1 + Z_2$

Parallel $\frac{1}{Z_{total}} = \frac{1}{Z_1} + \frac{1}{Z_2}$

Lets try an example using Ohm's law for capacitors. We have $V = IZ$

$$V = IZ = I \times \left(\frac{-j}{\omega C}\right)$$

Suppose we choose $I$ to be purely real (all this means is that we choose the phase of $I$ to be the reference phase, a purely imaginary current will fry you just as effectively as a purely real one!).
On the RHS (right hand side) we have two complex numbers to multiply.
Argand diagram





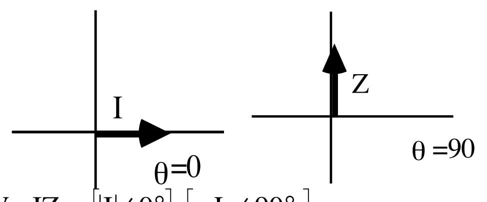



The voltage would be represented by a complex number (phasor) as follows:

$$V = IZ = \left[|I|\angle 0°\right] \times \left[\frac{1}{\omega C}\angle\text{-}90°\right]$$
$$= \frac{1}{\omega C}|I|\angle\left[0°\text{-}90°\right] = \frac{1}{\omega C}|I|\angle\text{-}90°$$

The interpretation of this is that the voltage lags behind the current by 90 degrees and that $|V| = \frac{1}{\omega C}|I|$

Similarly for an inductor: $V = j\omega LI$



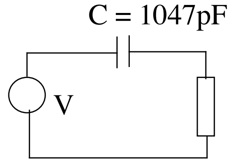

$$V = IZ = \left[|I|\angle 0°\right] \times \left[\omega L\angle 90°\right]$$
$$= \omega L|I|\angle\left[0°+90°\right] = \omega L|I|\angle 90°$$

The interpretation of this is that the voltage leads the current by 90 degrees and that $|V| = \omega L|I|$

Example: Lab setup: R and C in series



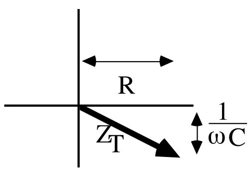





$Z_{Total} = Z_C + Z_R$

$Z_{Total} = \frac{-j}{\omega C} + R$

On an Argand diagram




the magnitude of this complex number is

$$|Z_{Total}| = \sqrt{\left(\frac{1}{\omega C}\right)^2 + R^2}$$

And the phase angle:

$$\text{Tan}\,\theta = \frac{\left(\frac{1}{\omega C}\right)}{R} = \frac{1}{\omega CR} = \frac{1}{2\pi fCR}$$

the voltage applied to the circuit is related to the current by ohms law V = IZ

$$V = IZ = I\left[R - \frac{j}{\omega C}\right]$$
$$= I\sqrt{R^2 + \frac{1}{\omega^2 C^2}}\angle\tan^{-1}(\omega CR)$$

Two results follow:

$$|V|^2 = |I|^2\left(R^2 + \frac{1}{\omega^2 C^2}\right)$$

but the voltage across the resistor is

$$|V_R|^2 = |I|^2\,|R|^2$$

and across the capacitor is

$$|V_C|^2 = |I|^2\left(\frac{1}{\omega^2 C^2}\right)$$

$$|V|^2 = |V_R|^2 + |V_C|^2$$

That means that for voltage magnitudes the total voltage (magnitude) is not the sum of the two voltage(magnitudes) ....

$$|V| \neq |V_R| + |V_C|$$

This arises essentially because $V_C$ and $V_R$ are not in phase.





The voltage lags behind the current by an angle $\phi$. For a purely resistive circuit V and I are in phase (otherwise not!).

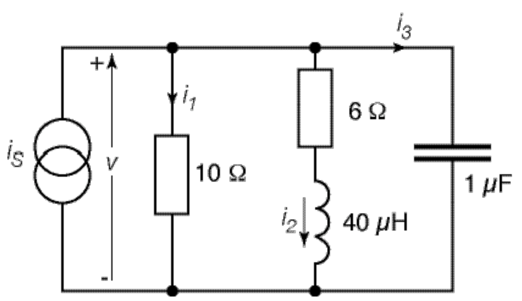
For a capacitive circuit V lags by 90 degrees, whereas for CR $\phi$ lies between 0 and -90. For an inductive circuit V leads by 90 degrees, whereas for LR $\phi$ lies between 0 and +90.





Example – Impedances in Series & Parallel

The sinusoidal current source in the circuit below is described as $i_S = 8\sin(200\,000\,t)$ Amps. Find the steady state expressions for the voltage across the source and the current in the three branches of the circuit $i_1$, $i_2$, $i_3$.

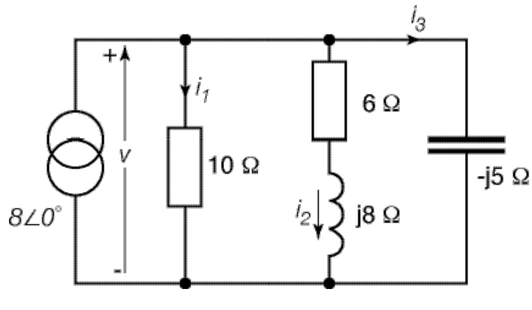



The current source can be written in phasor form as $\boldsymbol{I_S} = 8\angle 0° = 8 + j0$ amps as it is a pure sine wave with no phase shift. The currents $i_1$, $i_2$, and $i_3$, can be found by Ohm's law after we have calculated the voltage across the source. This voltage itself can be found once we know





the total combined impedance of the three branches.
First, recognising that $\omega$ = 200 000, the impedances of each parallel branch of the circuit above can be calculated, including,
$X_L = j\omega L = j \times 200\,000 \times 40\times10^{-6} = j8\ \Omega$
$X_C = -j/\omega C = -j/(200\,000 \times 1\times10^{-6}) = -j5\ \Omega$



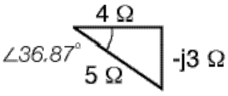

The impedances of each branch (from L to R) are 10 Ω, 6+$j$8 Ω and –$j$5 Ω respectively, so the total impedance $Z$ is given by,

$$\frac{1}{Z} = \frac{1}{10} + \frac{1}{6+j8} + \frac{1}{-j5} = \frac{1}{10} + \frac{1}{6+j8}\frac{6-j8}{6-j8} + \frac{1}{-j5}\frac{j}{j}$$

$$= \frac{10}{100} + \frac{6-j8}{36+64} + \frac{j20}{100} = \frac{16+j12}{100}$$

$$Z = \frac{100}{16+j12} = \frac{100}{16+j12}\frac{16-j12}{16-j12} = \frac{1600-j1200}{256+144}$$

so $Z = 4\text{-}j3$, or, considering this on a Argand diagram with $|Z| = \sqrt{4^2+3^2}$ and $\angle Z = \tan^{-1}(3/4)$




$Z = 4\text{-}j3 = 5\angle\text{-}36.87°$

Now the voltage across the source $\boldsymbol{V_S}$, is given by Ohm's law, $\boldsymbol{V_S} = \boldsymbol{I_S}Z$. Firstly using complex number notation:

$\boldsymbol{V_S} = 8+j0 \times 4\text{-}j3 = 32\text{-}j24$ volts,

which is equivalent to $\boldsymbol{V_S} = 40\angle\text{-}36.87°$ if translated to 'magnitude & phase' notation using an Argand diagram.
Actually, the calculation can be performed more easily in 'magnitude & phase' notation:

$\boldsymbol{V_S} = 8\angle 0° \times 5\angle\text{-}36.87° = 40\angle\text{-}36.87°$ volts,

(recalling that to multiply complex numbers, multiply the magnitudes and add the phases)





This voltage is labelled as $\boldsymbol{V_S}$ . The bold type indicates 'vector' and suggests that the number carries information on both the magnitude and phase. Although the actual voltage $v_s$ varies sinusoidally with time, the magnitude and phase are both steady, so we use an uppercase symbol to represent them. The value of $v_s$ can easily be obtained as we know the angular frequency of the circuit:

$v_s = 40 \sin (200\,000t - 36.87°)$ volts, or
$v_s = 40 \sin (200\,000t - 0.64)$ V all in radians

Now the question asks us to calculate the currents through each branch of the circuit. The voltage $\boldsymbol{V_S}$ is applied to each branch, so Ohm's law can be applied to each branch to calculate the currents currents $i_1$, $i_2$, and $i_3$, (or in the frequency domain $\boldsymbol{I_1}, \boldsymbol{I_2}, \boldsymbol{I_3}$)

$$I_1 = \frac{V_S}{Z_1} = \frac{32-j24}{10} = 3.2 - j2.4 = 4\angle -36.87°$$

With the 'magnitude & phase' result obtained from the complex vector result by an Argand diagram, and the results all in Amps.





This result could have been more easily obtained simply using 'magnitude & phase' notation (the phase of a resistance is 0°).

$$I_1 = \frac{40\angle -36.87°}{10\angle 0°} = 4\angle -36.87°$$

Recall that phasor division is performed by dividing magnitudes and subtracting phases. The time dependant current $i_1$ is easily obtained from information on the magnitude and phase of the signal:

$i_1 = 4 \sin (200\,000t - 36.87°)$

Similarly for the second branch,

$$I_2 = \frac{V_S}{Z_2} = \frac{32-j24}{6+j8} = \frac{32-j24}{6+j8}\frac{6-j8}{6-j8}$$
$$= \frac{192-192-j144-j256}{36+64} = -j4 \ Amps$$

and as $6+j8 = 10\angle 53.13°$ by Argand diagram,

$$I_2 = \frac{V_S}{Z_2} = \frac{40\angle -36.87°}{10\angle 53.13°} = \frac{40}{10}\angle -36.87° - 53.13°$$

$I_2 = 4\angle\text{-}90°$, which is of course equal to $-j4$ A





The time dependant current $i_2$ is again obtained from information on the magnitude and phase of the signal:

$i_2 = 4 \sin (200\,000t - 90°)$

Finally, similar calculations can be made for the third branch of the system:

$$I_3 = \frac{V_S}{Z_3} = \frac{32-j24}{-j5} = \frac{32-j24}{-j5}\frac{j5}{j5} = 4.8 + j6.4$$

or as $-j5 = 5\angle -90°$,

$$I_3 = \frac{V_S}{Z_3} = \frac{40\angle -36.87°}{5\angle -90°} = \frac{40}{5}\angle -36.87° + 90°$$

$I_3 = 8\angle 53.13°$, which is equal to $4.8+j6.4$ A
The time dependant current $i_3$ is:

$i_3 = 8 \sin (200\,000t + 53.13°)$

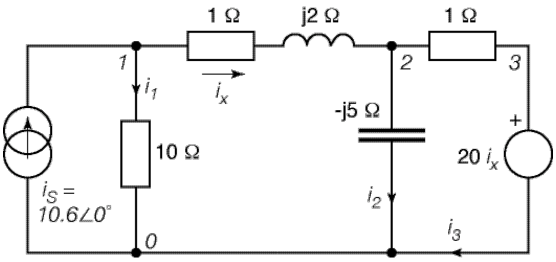
Summing $\boldsymbol{I_1}, \boldsymbol{I_2}, \boldsymbol{I_3}$ is best done with complex numbers:

$\boldsymbol{I_1}+\boldsymbol{I_2}+\boldsymbol{I_3} = 3.2\text{-}j2.4 + \text{-}j4 + 4.8+j6.4 = 8.0$ A

which is equal, as required, to the magnitude and phase of the original $i_S$.

Example – Nodal Analysis

Calculate the currents $i_1$, $i_2$, and $i_3$ in the following circuit.




The existence of the voltage source between nodes 0 and 3 makes it difficult to apply the principles of parallel and serial circuits. In addition, the voltage source is part of an active / amplifying circuit – its value depends on the current $i_x$. To obtain a solution nodal analysis is applied to nodes 1 and 2.

Node 1 $$-10.6 + \frac{V_1}{10} + \frac{V_1 - V_2}{1 + j2} = 0$$

multiply by 1+$j$2 and collect terms:

$$-10.6(1 + j2) + V_1 - V_2 + 0.1V_1(1 + j2) = 0$$





$$V_1(1.1 + j0.2) - V_2 = 10.6 + j21.2 \qquad (1)$$

Node 2 $$\frac{V_2 - V_1}{1 + j2} + \frac{V_2}{-j5} + \frac{V_2 - 20I_X}{5} = 0$$

multiply by 1+$j$2 and substitute $I_X = \frac{V_1 - V_2}{1 + j2}$

$$V_2 - V_1 + \frac{(1 + j2)V_2}{-j5} + \frac{(1 + j2)}{5}\left[V_2 - \frac{20(V_1 + V_2)}{1 + j2}\right] = 0$$

$$V_2 - V_1 + 0.2(j - 2)V_2 + \tfrac{0.2(1 + j2)}{5}V_2 - 4V_1 + 4V_2 = 0$$

$$-5V_1 + (4.8 + j0.6)V_2 = 0 \qquad (2)$$

To solve (1) and (2) multiply (1) by 4.8+$j$0.6 and add.

(1) $V_1(1.1 + j0.2)(4.8 + j0.6) - (4.8 + j0.6)V_2 = (4.8 + j0.6)(10.6 - j21.2)$

(2) $-5V_1 + (4.8 + j0.6)V_2 = 0$

$$\Rightarrow \quad V_1[(1.1 + j0.2)(4.8 + j0.6) - 5] = (4.8 + j0.6)(10.6 - j21.2)$$

$$\Rightarrow \quad V_1 = \frac{(4.8 + j0.6)(10.6 - j21.2)}{[(1.1 + j0.2)(4.8 + j0.6) - 5]} = \frac{38.16 + j108.12}{0.16 - j1.62}$$

$$\Rightarrow \quad V_1 = \frac{38.16 + j108.12}{0.16 + j1.62}\frac{0.16 - j1.62}{0.16 - j1.62} = 68.4 - j16.8 \text{ volts}$$

Similarly, substituting back into (2)

(2) $-5(68.4 - j16.8) + (4.8 + j0.6)V_2 = 0$

$$\Rightarrow \quad V_2 = \frac{5(68.4 + j16.8)}{4.8 + j0.6} = \frac{5(68.4 + j16.8)}{4.8 + j0.6}\frac{4.8 - j0.6}{4.8 - j0.6} = 68 - j26 \text{ volts}$$

so $V_1$ = 68.4-$j$16.8 volts and $V_2$ = 68-$j$26 volts





From these voltages all the currents can be calculated:

$$I_1 = \frac{V_1}{10} = \frac{68.4 - j16.8}{10} = 6.84 - j1.68 \text{ amps}$$

$$I_2 = \frac{V_2}{-j5} = \frac{68 - j26}{-j5} = \frac{68 - j26}{-j5} = 5.2 + j13.6 \text{ amps}$$

$$I_X = \frac{V_1 - V_2}{1 + j2} = \frac{68.4 - j16.8 - 68 + j26}{1 + j2} = \frac{0.4 + j9.2}{1 + j2}$$
$$= \frac{0.4 + j9.2}{1 + j2}\frac{1 - j2}{1 - j2} = \frac{18.8 + j8.4}{5} = 3.76 + j1.68$$

$$I_3 = \frac{V_2 - 20i_X}{5} = \frac{68 - j26 - 20(3.76 + j1.68)}{5}$$
$$= 13.6 - j5.2 - 4(3.76 + j1.68) = -1.44 - j11.92$$

As a check, we expect $\boldsymbol{I_1} + \boldsymbol{I_X} = \boldsymbol{I_S}$ and $\boldsymbol{I_2}+\boldsymbol{I_3} = \boldsymbol{I_X}$:

$\boldsymbol{I_1}+\boldsymbol{I_X}$ = 6.84-j1.68 + 3.76+j1.68 = 10.6 Amps
$\boldsymbol{I_2}+\boldsymbol{I_3}$ = -1.44-j11.92 + 5.2+j13.6 = 3.76+j1.68 A