

ON SURFACE WAVES IN A FINITELY DEFORMED MAGNETOELASTIC HALF-SPACE

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Rayleigh-type surface waves propagating in an incompressible isotropic half-space of nonconducting magnetoelastic material are studied for a half-space subjected to a finite pure homogeneous strain and a uniform magnetic field. First, the equations and boundary conditions governing linearized incremental motions superimposed on an initial motion and underlying electromagnetic field are derived and then specialized to the quasimagneto-static approximation. The magnetoelastic material properties are characterized in terms of a “total” isotropic energy density function that depends on both the deformation and a Lagrangian measure of the magnetic induction. The problem of surface wave propagation is then analyzed for different directions of the initial magnetic field and for a simple constitutive model of a magnetoelastic material in order to evaluate the combined effect of the finite deformation and magnetic field on the surface wave speed. It is found that a magnetic field in the considered (sagittal) plane in general destabilizes the material compared with the situation in the absence of a magnetic field, and a magnetic field applied in the direction of wave propagation is more destabilizing than that applied perpendicular to it.

Keywords: Nonlinear magnetoelasticity; magnetoacoustics; surface waves; finite deformation.

1. Introduction

Coupling of electromagnetic and mechanical phenomena in continuous media has received considerable attention in the recent literature due to the development of electro- and magneto-sensitive elastomers and other polymers for use as “smart materials”; see, for example, Jolly *et al.* [1996], Ginder *et al.* [2002], Lokander and Stenberg [2003], Yalcintas and Dai [2004], Varga *et al.* [2006], Boczowska and Awitjan [2009]. The problem of wave propagation under a state of finite deformation in the presence of an electromagnetic field is, in particular, very important for various applications, such as nondestructive evaluation through electromagnetic acoustic

transducers [Shapoorabadi *et al.*, 2005]. The effect of initial stress on the propagation of magnetoelastic waves was addressed as early as 1966 by Yu and Tang [1966], who considered the propagation of plane harmonic waves for some special cases of initial stress relevant to seismic wave propagation. Two papers by De and Sengupta [1971, 1972] used the equations from Yu and Tang [1966] in order to discuss surface and interfacial waves in magnetoelastic conducting solids. However, it has been noted elsewhere [Destrade and Ogden, 2011a] that the formulation of the equations of a material with initial stress, discussed in more detail in the paper by Tang [1967], is incorrect.

A paper by Maugin [1981] reviewed the major developments in deformable magnetoelastic materials up to that date with special emphasis on wave propagation in magnetizable conducting materials. This was followed by a series of works, notably those by Maugin and Hakmi [1985] on magnetoelastic surface waves with a bias magnetic field orthogonal to the sagittal plane, Abd-Alla and Maugin [1987] on the general form of the magnetoacoustic equations, Abd-Alla and Maugin [1988] on magnetoelastic waves in anisotropic materials, Lee and Its [1992] on Rayleigh waves in an undeformed magnetoelastic conductor and Hefni *et al.* [1995a, 1995b, 1995c] on surface and bulk magnetoelastic waves in electrical conductors. Most of the work in this field is based on the study of electromagnetic phenomena in continua by Pao [1978], Maugin [1988] and Eringen and Maugin [1990a, 1990b]. Although our main concern in the present paper is with magnetoelastic waves, the parallel development of the theory for waves in electroelastic materials should also be mentioned. Relevant references include Baumhauer and Tiersten [1973], Nelson [1979], Sinha and Tiersten [1979], Pouget and Maugin [1981], Maugin *et al.* [1992], Tiersten [1995], Chai and Wu [1996], Yang [2001], Simionescu-Panait [2002], Liu *et al.* [2003], Yang and Hu [2004], Hu *et al.* [2004] and Dorfmann and Ogden [2010].

Recently, a new constitutive formulation based on a “total” energy density function was developed by Dorfmann and Ogden [2004], wherein the solutions of some basic boundary-value problems were obtained using two alternative forms of the energy density with different independent magnetic vectors; see also Dorfmann and Ogden [2005] for the discussion of further boundary-value problems. In the paper by Otténio *et al.* [2008], which was based on the formulation of Dorfmann and Ogden [2004], the equations governing time-independent linearized incremental deformations and magnetic fields superimposed on a static finite deformation and magnetic field were derived. These were then applied to analyze the effect of the presence of a magnetic field normal to the half-space boundary on the stability of a deformed magnetoelastic half-space.

In the present paper, we build upon the work of Otténio *et al.* [2008] by analyzing surface waves on a half-space of an incompressible isotropic magnetoelastic material subjected to a homogeneous finite deformation and a uniform magnetic field. In Sec. 2, we first summarize the basic continuum kinematics and the governing equations and boundary conditions of continuum electromagnetodynamics

in both Eulerian and Lagrangian forms, and the notations to be used in this paper. Section 3 follows the formulation of the incremental equations of electrodynamics superimposed on a finite motion in the presence of an electromagnetic field summarized in Ogden [2009], and the equations are then specialized to the case of the quasimagnetostatic approximation.

In Sec. 4, the constitutive equations for both compressible and incompressible isotropic magnetoelastic solids are introduced based on a “total” energy density that is expressed in terms of (six and five, respectively) independent invariants that combine the deformation and a Lagrangian form of the magnetic induction vector. This is then used to derive the magnetoelastic “moduli” tensors and the governing equations explicitly. The Cartesian components of these tensors, referred to the principal axes of the underlying deformation, are summarized in Appendix B for ease of reference. The focus of the remainder of the paper is then on incompressible materials. Following Destrade and Ogden [2011b], a brief account of homogeneous plane waves propagating in an infinite space subject to a homogeneous deformation and a uniform magnetic field is provided in Sec. 5 in order to derive a generalized form of the strong ellipticity condition appropriate for magnetoelastic materials that is used subsequently.

In Sec. 6, the general equations from Sec. 3 are specialized to two dimensions in order to consider plane motions in a principal plane of an underlying pure homogeneous strain with the magnetic field lying parallel to the considered (sagittal) plane. Then, in Sec. 7, these equations, along with appropriate boundary conditions, are applied in consideration of surface waves propagating parallel to the boundary of a deformed half-space, with the displacement and incremental magnetic field components restricted to the sagittal plane. Two special cases of the magnetic field direction are examined in detail by the way of illustration — a magnetic field parallel to the direction of propagation and the one normal to the direction within the sagittal plane. We then show that on taking the magnetic field to be zero the equations reduce to those obtained for a purely elastic material [Dowaikh and Ogden, 1990].

For each magnetic field direction numerical results are obtained based on a simple, so-called Mooney–Rivlin magnetoelastic material model in order to illustrate the dependence of the surface wave speed on the magnitude of the magnetic induction (as well as the underlying deformation). It is found that in general the magnetic field compromises the mechanical stability of the half-space, although for some values of the magnetoelastic coupling parameters small values of the magnetic induction stabilize the half-space, but this effect is reversed as the magnitude of the magnetic induction increases further. This applies to a magnetic induction perpendicular to or in the direction of propagation within the sagittal plane, but a magnetic induction parallel to the direction of propagation has a stronger influence than that is perpendicular to it.

It is also shown, in Sec. 8, that if the magnetic field is perpendicular to the sagittal plane then, for the considered material model, it has no influence on the

surface wave speed, although for other possible models this will not in general be the case. Finally, Sec. 9 contains some brief concluding remarks.

2. Basic Equations

2.1. Kinematics

The undeformed stress-free *reference configuration* of a continuous body is denoted by \mathcal{B}_r and its boundary by $\partial\mathcal{B}_r$. Let \mathcal{B}_t , the *current configuration*, be the region occupied by the body at time t and $\partial\mathcal{B}_t$ its boundary. The material points in \mathcal{B}_r are identified by the position vector \mathbf{X} which becomes the position vector \mathbf{x} in \mathcal{B}_t . The time-dependent deformation (or motion) of the body is described by the invertible mapping χ , with $\mathbf{x} = \chi(\mathbf{X}, t)$, and χ and its inverse are assumed to be sufficiently regular in space and time. The velocity \mathbf{v} and acceleration \mathbf{a} of a material particle \mathbf{X} are defined by

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{x}_{,t} = \frac{\partial}{\partial t}\chi(\mathbf{X}, t), \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{v}_{,t} = \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\chi(\mathbf{X}, t), \quad (2.1)$$

where the subscript t following a comma denotes the material time derivative.

Throughout this paper, grad, div, curl denote the standard differential operators with respect to \mathbf{x} , and Grad, Div, Curl denote the corresponding operators with respect to \mathbf{X} .

The *deformation gradient tensor* is defined as $\mathbf{F} = \text{Grad}\chi(\mathbf{X}, t)$ and its determinant is denoted by $J = \det \mathbf{F}$, with $J > 0$. For an incompressible material the constraint

$$J \equiv \det \mathbf{F} = 1 \quad (2.2)$$

has to be satisfied. Associated with \mathbf{F} are the left and right Cauchy–Green tensors, defined by

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{c} = \mathbf{F}^T\mathbf{F}, \quad (2.3)$$

respectively.

Let $\mathbf{\Gamma} = \text{grad}\mathbf{v}$ denote the velocity gradient, tr and T the trace and transpose of a second-order tensor, respectively, $\mathbf{0}$ the zero vector and \mathbf{O} the second-order zero tensor. Then the following standard kinematic identities are noted:

$$\mathbf{F}_{,t} = \mathbf{\Gamma}\mathbf{F}, \quad (2.4a)$$

$$(\mathbf{F}^{-1})_{,t} = -\mathbf{F}^{-1}\mathbf{\Gamma}, \quad (2.4b)$$

$$J_{,t} = J \text{tr}\mathbf{\Gamma} = J \text{div}\mathbf{v}, \quad (2.4c)$$

$$\text{Div}(J\mathbf{F}^{-1}) = \mathbf{0}, \quad (2.4d)$$

$$\text{div}(J^{-1}\mathbf{F}) = \mathbf{0}, \quad (2.4e)$$

$$\text{Curl}(\mathbf{F}^T) = \mathbf{O}, \tag{2.4f}$$

$$\text{curl}(\mathbf{F}^{-T}) = \mathbf{O}. \tag{2.4g}$$

For an incompressible material $\text{div } \mathbf{v} = 0$.

2.2. Equations of electromagnetism

Let $\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}, \mathbf{J}, \mathbf{K}, \rho_e$ and σ_e be the electric field, electric displacement, magnetic induction, magnetic field, volume current density, surface current density, volume charge density and surface charge density, respectively. It should be noted that the volume current density \mathbf{J} is different from $J = \det \mathbf{F}$ defined in the above subsection. We work within the nonrelativistic framework, with Maxwell’s equations of electromagnetism given in Eulerian form by

$$\text{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad \text{div} \mathbf{D} = \rho_e, \quad \text{div} \mathbf{B} = 0, \tag{2.5}$$

with the boundary conditions

$$\mathbf{n} \times \llbracket \mathbf{E} + \mathbf{v} \times \mathbf{B} \rrbracket = \mathbf{0}, \tag{2.6a}$$

$$\mathbf{n} \cdot \llbracket \mathbf{D} \rrbracket = \sigma_e, \tag{2.6b}$$

$$\mathbf{n} \times \llbracket \mathbf{H} - \mathbf{v} \times \mathbf{D} \rrbracket = \mathbf{K} - \sigma_e \mathbf{v}_s, \tag{2.6c}$$

$$\mathbf{n} \cdot \llbracket \mathbf{B} \rrbracket = 0 \tag{2.6d}$$

on $\partial \mathcal{B}_t$, where \mathbf{n} is the unit outward normal to $\partial \mathcal{B}_t$ and \mathbf{v}_s is the value of \mathbf{v} on $\partial \mathcal{B}_t$. Here $\llbracket \mathbf{a} \rrbracket$ represents the jump in vector \mathbf{a} across the boundary in the sense $\llbracket \mathbf{a} \rrbracket = \mathbf{a}^o - \mathbf{a}^i$, where the superscripts “o” and “i” signify “outside” and “inside”, respectively, and surface polarization is not included.

Lagrangian forms of the physical quantities in (2.5) are defined by (see, e.g., Maugin [1988] and Ogden [2009])

$$\begin{aligned} \mathbf{D}_l &= J\mathbf{F}^{-1}\mathbf{D}, & \mathbf{E}_l &= \mathbf{F}^T\mathbf{E}, & \mathbf{H}_l &= \mathbf{F}^T\mathbf{H}, & \mathbf{B}_l &= J\mathbf{F}^{-1}\mathbf{B}, \\ \mathbf{J}_E &= J\mathbf{F}^{-1}(\mathbf{J} - \rho_e\mathbf{v}), & \rho_E &= J\rho_e. \end{aligned} \tag{2.7}$$

Using these relations and following the analysis in Ogden [2009], we can rewrite Maxwell’s equations in Lagrangian form as

$$\text{Curl}(\mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l) = -\mathbf{B}_{l,t}, \quad \text{Div} \mathbf{D}_l = \rho_E, \tag{2.8}$$

$$\text{Curl}(\mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l) = \mathbf{D}_{l,t} + \mathbf{J}_E, \quad \text{Div} \mathbf{B}_l = 0, \tag{2.9}$$

where $\mathbf{V} = \mathbf{F}^{-1}\mathbf{v}$, along with the boundary conditions

$$\mathbf{N} \times \llbracket \mathbf{E}_l + \mathbf{V} \times \mathbf{B}_l \rrbracket = \mathbf{0}, \quad \mathbf{N} \cdot \llbracket \mathbf{D}_l \rrbracket = \sigma_E, \tag{2.10}$$

$$\mathbf{N} \times \llbracket \mathbf{H}_l - \mathbf{V} \times \mathbf{D}_l \rrbracket = \mathbf{K}_l - \sigma_E \mathbf{V}_s, \quad \mathbf{N} \cdot \llbracket \mathbf{B}_l \rrbracket = 0 \tag{2.11}$$

on $\partial\mathcal{B}_r$. The transformation from (2.6a)–(2.6d) to (2.10) and (2.11) requires use of Nanson’s formula $\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA$ connecting reference and current area elements dA and da , where \mathbf{N} is the unit outward normal to $\partial\mathcal{B}_r$. Here each term is evaluated on $\partial\mathcal{B}_r$, \mathbf{V}_s is the value of \mathbf{V} on the boundary, \mathbf{K}_l is the surface current density per unit area of $\partial\mathcal{B}_r$ given by $\mathbf{K}_l = \mathbf{F}^{-1}\mathbf{K}da/dA$ and $\sigma_E = \sigma_e da/dA$ is the surface charge density per unit area of $\partial\mathcal{B}_r$.

2.3. Continuum electromagnetodynamic equations

In Eulerian form, the linear momentum balance equation may be written as

$$\operatorname{div}\boldsymbol{\tau} + \rho\mathbf{f} = \rho\mathbf{a}, \tag{2.12}$$

where ρ is the mass density, \mathbf{f} is the mechanical body force density per unit mass and $\boldsymbol{\tau}$ is the so-called *total Cauchy stress tensor*, which incorporates the electromagnetic body forces. In Lagrangian form, the equation of motion is

$$\operatorname{Div}\mathbf{T} + \rho_r\mathbf{f} = \rho_r\mathbf{a}, \tag{2.13}$$

where \mathbf{T} is the total nominal stress tensor and ρ_r is the reference mass density, and we note the connections

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\mathbf{T}, \quad \rho_r = \rho J. \tag{2.14}$$

The transformation from (2.12) to (2.13) is effected by use of (2.4e).

If there are no intrinsic mechanical couples, which is assumed to be the case, then, by virtue of the definition of the total stress, the electric and magnetic couples are absorbed in such a way that $\boldsymbol{\tau}$ is *symmetric*. The angular momentum balance equation is then expressed in either of the equivalent forms

$$\boldsymbol{\tau}^T = \boldsymbol{\tau}, \quad (\mathbf{F}\mathbf{T})^T = \mathbf{F}\mathbf{T}. \tag{2.15}$$

On any part of the boundary $\partial\mathcal{B}_r$ where the traction is prescribed the boundary condition may be given as

$$\mathbf{T}^T\mathbf{N} = \mathbf{t}_A + \mathbf{t}_M, \tag{2.16}$$

where \mathbf{t}_A and \mathbf{t}_M are the Lagrangian representations of the mechanical and magnetic contributions to the traction per unit area on the boundary $\partial\mathcal{B}_r$.

3. Incremental Equations

On the initial motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, we superimpose an incremental motion given by

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t), \tag{3.1}$$

where here and henceforth incremented quantities are denoted by a superimposed dot. The Eulerian counterpart of $\dot{\mathbf{x}}$ is the displacement $\mathbf{u}(\mathbf{x}, t) = \dot{\boldsymbol{\chi}}(\mathbf{X}, t)$. Then, an

increment in the velocity \mathbf{v} is given by

$$\dot{\mathbf{v}} = \dot{\mathbf{x}}_{,t} = \mathbf{u}_{,t}. \tag{3.2}$$

We define $\mathbf{L} = \text{grad } \mathbf{u}$ and then obtain the useful relations

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \tag{3.3a}$$

$$\dot{\mathbf{F}}_{,t} = (\text{grad } \dot{\mathbf{v}})\mathbf{F}, \tag{3.3b}$$

which supplement those in (2.4a)–(2.4g).

For an incompressible material, the constraint $J = 1$ leads to

$$\text{div } \mathbf{v} = 0, \tag{3.4a}$$

$$\text{div } \mathbf{u} = 0, \tag{3.4b}$$

$$\text{div } \dot{\mathbf{v}} = 0, \tag{3.4c}$$

the first of which is exact while the second and third are linear approximations.

On taking increments of the Maxwell equations (2.8) and (2.9), we obtain

$$\text{Curl}(\dot{\mathbf{E}}_l + \mathbf{V} \times \dot{\mathbf{B}}_l + \dot{\mathbf{V}} \times \mathbf{B}_l) = -\dot{\mathbf{B}}_{l,t}, \quad \text{Div } \dot{\mathbf{D}}_l = \dot{\rho}_E, \tag{3.5}$$

$$\text{Curl}(\dot{\mathbf{H}}_l - \mathbf{V} \times \dot{\mathbf{D}}_l - \dot{\mathbf{V}} \times \mathbf{D}_l) = \dot{\mathbf{D}}_{l,t} + \dot{\mathbf{J}}_E, \quad \text{Div } \dot{\mathbf{B}}_l = 0, \tag{3.6}$$

and from the mechanical balance equations (2.13) and (2.15), we have

$$\text{Div } \dot{\mathbf{T}} + \rho_r \dot{\mathbf{f}} = \rho_r \mathbf{u}_{,tt}, \quad \mathbf{L}\mathbf{F}\mathbf{T} + \mathbf{F}\dot{\mathbf{T}} = \mathbf{T}^T \mathbf{F}^T \mathbf{L}^T + \dot{\mathbf{T}}^T \mathbf{F}^T, \tag{3.7}$$

wherein use has been made of (3.3a).

Analogously to Eq. (2.7), we define updated (i.e., pushed-forward) forms of the increments $\dot{\mathbf{T}}, \dot{\mathbf{B}}_l, \dot{\mathbf{D}}_l, \dot{\mathbf{E}}_l, \dot{\mathbf{H}}_l$ as

$$\begin{aligned} \dot{\mathbf{T}}_0 &= J^{-1} \mathbf{F} \dot{\mathbf{T}}, & \dot{\mathbf{B}}_{l0} &= J^{-1} \mathbf{F} \dot{\mathbf{B}}_l, & \dot{\mathbf{D}}_{l0} &= J^{-1} \mathbf{F} \dot{\mathbf{D}}_l, \\ \dot{\mathbf{E}}_{l0} &= \mathbf{F}^{-T} \dot{\mathbf{E}}_l, & \dot{\mathbf{H}}_{l0} &= \mathbf{F}^{-T} \dot{\mathbf{H}}_l, \end{aligned} \tag{3.8}$$

where the subscript 0 is used to indicate the push-forward operation. We use these push-forward forms to update the incremented governing equations to obtain

$$\text{curl}(\dot{\mathbf{E}}_{l0} + \mathbf{v} \times \dot{\mathbf{B}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l,t0}, \quad \text{div } \dot{\mathbf{D}}_{l0} = \dot{\rho}_{E0}, \tag{3.9}$$

$$\text{curl}(\dot{\mathbf{H}}_{l0} - \mathbf{v} \times \dot{\mathbf{D}}_{l0} - \dot{\mathbf{v}} \times \mathbf{D}) = \dot{\mathbf{D}}_{l,t0} + \dot{\mathbf{J}}_{E0}, \quad \text{div } \dot{\mathbf{B}}_{l0} = 0 \tag{3.10}$$

and

$$\text{div } \dot{\mathbf{T}}_0 + \rho_r \dot{\mathbf{f}} = \rho_r \mathbf{u}_{,tt}, \quad \mathbf{L}\boldsymbol{\tau} + \dot{\mathbf{T}}_0 = \boldsymbol{\tau} \mathbf{L}^T + \dot{\mathbf{T}}_0^T. \tag{3.11}$$

It should be noted that the push-forward and material time derivative operations do not in general commute. However, in the special case of $\mathbf{v} = \mathbf{0}$, they do and then $\dot{\mathbf{B}}_{l0,t} = \dot{\mathbf{B}}_{l,t0}$ and $\dot{\mathbf{D}}_{l0,t} = \dot{\mathbf{D}}_{l,t0}$.

3.1. The quasimagnetostatic approximation

We now consider the initial configuration to be purely static and subject only to magnetic and mechanical effects, with $\mathbf{E} = \mathbf{0}$, $\mathbf{D} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ and no mechanical body forces ($\mathbf{f} = \mathbf{0}$). We assume that there are no volume or surface charges or currents, so that $\rho_e = \sigma_e = 0$ and $\mathbf{J} = \mathbf{0}$, while \mathbf{H} and \mathbf{B} are independent of time. Additionally, we consider a nonconducting material so that $\dot{\mathbf{J}}_{l0} = \mathbf{0}$. The updated incremented governing equations then specialize to

$$\text{curl}(\dot{\mathbf{E}}_{l0} + \dot{\mathbf{v}} \times \mathbf{B}) = -\dot{\mathbf{B}}_{l0,t}, \quad \text{div} \dot{\mathbf{D}}_{l0} = 0, \tag{3.12}$$

$$\text{curl} \dot{\mathbf{H}}_{l0} = \dot{\mathbf{D}}_{l,t0}, \quad \text{div} \dot{\mathbf{B}}_{l0} = 0 \tag{3.13}$$

and

$$\text{div} \dot{\mathbf{T}}_0 = \rho_r \mathbf{u}_{,tt}. \tag{3.14}$$

We now focus on the magnetoacoustic (or quasimagnetostatic) approximation of the equations, which allows the incremental electric field and displacement to be neglected. It can be shown that they are of order v/c ($\ll 1$) times the retained terms in the equations, where c is the speed of electromagnetic waves in vacuum and v is a typical magnitude of the acoustic wave speed. After the approximation is applied, the remaining equations, coupling magnetic and mechanical effects, are

$$\text{curl} \dot{\mathbf{H}}_{l0} = \mathbf{0}, \quad \text{div} \dot{\mathbf{B}}_{l0} = 0, \quad \text{div} \dot{\mathbf{T}}_0 = \rho_r \mathbf{u}_{,tt}. \tag{3.15}$$

These are the equations we use in the rest of the paper for the interior of the material.

Outside the material, which may be vacuum or a nonmagnetizable (and nonpolarizable) material we use a superscript $*$ to indicate field quantities. Thus, \mathbf{H}^* and \mathbf{B}^* , respectively, are the magnetic field and magnetic induction, which are in the simple relation $\mathbf{B}^* = \mu_0 \mathbf{H}^*$, where μ_0 is the vacuum permeability. Then the magnetostatic equations are

$$\text{div} \mathbf{B}^* = 0, \quad \text{curl} \mathbf{H}^* = \mathbf{0}, \tag{3.16}$$

and in the quasimagnetostatic approximation their incremental counterparts are

$$\text{div} \dot{\mathbf{B}}^* = 0, \quad \text{curl} \dot{\mathbf{H}}^* = \mathbf{0}, \tag{3.17}$$

with $\dot{\mathbf{B}}^* = \mu_0 \dot{\mathbf{H}}^*$.

Henceforth, we use the notations \mathcal{B} and $\partial\mathcal{B}$ for the (time-independent) initial configuration upon which the infinitesimal motion is superimposed.

3.2. Incremental boundary conditions

The boundary condition (2.6d) for the magnetic induction is written $(\mathbf{B} - \mathbf{B}^*) \cdot \mathbf{n} = 0$ on $\partial\mathcal{B}$. Since there is no deformation outside the material (in the case that it is a vacuum, which we assume henceforth) there is no physical meaning attached to

a Lagrangian form of the magnetic induction, so when the boundary condition is expressed in Lagrangian form it becomes

$$(\mathbf{B}_l - J\mathbf{F}^{-1}\mathbf{B}^*) \cdot \mathbf{N} = 0 \quad \text{on } \partial\mathcal{B}_r. \tag{3.18}$$

On taking an increment of the above equation, and then updating and using the incompressibility condition (3.4b) we obtain the incremental boundary condition

$$(\dot{\mathbf{B}}_{l0} - \dot{\mathbf{B}}^* + \mathbf{L}\mathbf{B}^*) \cdot \mathbf{n} = 0 \quad \text{on } \partial\mathcal{B}. \tag{3.19}$$

The corresponding incremental form of the boundary condition (2.6c) for the magnetic field, with $\mathbf{K} = \mathbf{0}$, now becomes

$$(\dot{\mathbf{H}}_{l0} - \mathbf{L}^T\mathbf{H}^* - \dot{\mathbf{H}}^*) \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\mathcal{B}. \tag{3.20}$$

In order to arrive at the corresponding incremental traction boundary condition we need to define the Maxwell stress outside the material, denoted by $\boldsymbol{\tau}^*$. This is symmetric and given by

$$\boldsymbol{\tau}^* = \mu_0^{-1} \left[\mathbf{B}^* \otimes \mathbf{B}^* - \frac{1}{2}(\mathbf{B}^* \cdot \mathbf{B}^*)\mathbf{I} \right], \tag{3.21}$$

where \mathbf{I} is the identity tensor. The incremental Maxwell stress is then obtained as

$$\dot{\boldsymbol{\tau}}^* = \mu_0^{-1} [\dot{\mathbf{B}}^* \otimes \mathbf{B}^* + \mathbf{B}^* \otimes \dot{\mathbf{B}}^* - (\dot{\mathbf{B}}^* \cdot \mathbf{B}^*)\mathbf{I}]. \tag{3.22}$$

The Lagrangian form of the Maxwell stress is $J\mathbf{F}^{-1}\boldsymbol{\tau}^*$, which is defined only on the boundary \mathcal{B}_r , and the magnetic contribution \mathbf{t}_M to the traction on \mathcal{B}_r in (2.16) is given by

$$\mathbf{t}_M = J\boldsymbol{\tau}^*\mathbf{F}^{-T}\mathbf{N} \quad \text{on } \partial\mathcal{B}_r. \tag{3.23}$$

On taking an increment of this equation, we obtain

$$\dot{\mathbf{t}}_M = J\dot{\boldsymbol{\tau}}^*\mathbf{F}^{-T}\mathbf{N} - J\boldsymbol{\tau}^*\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{N} + J(\text{div } \mathbf{u})\boldsymbol{\tau}^*\mathbf{F}^{-T}\mathbf{N}, \tag{3.24}$$

which on pushing forward and using the incompressibility condition (3.4b), gives

$$\dot{\mathbf{t}}_{M0} = \dot{\boldsymbol{\tau}}^*\mathbf{n} - \boldsymbol{\tau}^*\mathbf{L}^T\mathbf{n} \quad \text{on } \partial\mathcal{B}. \tag{3.25}$$

When there is also a mechanical traction \mathbf{t}_A , with increment $\dot{\mathbf{t}}_A$, the traction boundary condition is written as

$$\dot{\mathbf{T}}_0^T\mathbf{n} = \dot{\mathbf{t}}_{A0} + \dot{\mathbf{t}}_{M0} \tag{3.26}$$

at any point of $\partial\mathcal{B}$ where the traction is prescribed.

4. Constitutive Relations

Following Dorfmann and Ogden [2004, 2005], we consider a magnetoelastic material for which the constitutive law is given in terms of a total potential energy function, $\Omega = \Omega(\mathbf{F}, \mathbf{B}_l)$, defined per unit reference volume. This yields the simple formulas

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{4.1}$$

for the total nominal stress and the Lagrangian magnetic field. Their Eulerian counterparts are

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}}, \quad \mathbf{H} = \mathbf{F}^{-T} \frac{\partial \Omega}{\partial \mathbf{B}_l}. \tag{4.2}$$

In the case of an incompressible material, we have the constraint $J = 1$ and the above equations for the stresses are modified to

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{F}^{-1}, \quad \boldsymbol{\tau} = \mathbf{F} \frac{\partial \Omega}{\partial \mathbf{F}} - p \mathbf{I}, \tag{4.3}$$

where p is a Lagrange multiplier associated with the constraint and \mathbf{I} is again the identity tensor.

For an isotropic magnetoelastic material, Ω can be expressed in terms of six independent scalar invariants of $\mathbf{c} = \mathbf{F}^T \mathbf{F}$ and $\mathbf{B}_l \otimes \mathbf{B}_l$. One possible set of invariants, is

$$I_1 = \text{tr } \mathbf{c}, \quad I_2 = \frac{1}{2}[(\text{tr } \mathbf{c})^2 - \text{tr}(\mathbf{c}^2)], \quad I_3 = \det \mathbf{c} = J^2, \tag{4.4}$$

$$I_4 = \mathbf{B}_l \cdot \mathbf{B}_l, \quad I_5 = (\mathbf{c} \mathbf{B}_l) \cdot \mathbf{B}_l, \quad I_6 = (\mathbf{c}^2 \mathbf{B}_l) \cdot \mathbf{B}_l. \tag{4.5}$$

We adopt these here and confine our attention to isotropic magnetoelastic materials.

The total nominal stress and the Lagrangian magnetic field can then be expanded in the forms

$$\mathbf{T} = \sum_{i \in \mathcal{I}} \Omega_i \frac{\partial I_i}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \sum_{i \in \mathcal{J}} \Omega_i \frac{\partial I_i}{\partial \mathbf{B}_l}, \tag{4.6}$$

where $\Omega_i = \partial \Omega / \partial I_i$, $i = 1, \dots, 6$, \mathcal{I} is the set $\{1, 2, 3, 5, 6\}$, or $\{1, 2, 5, 6\}$ for an incompressible material, and \mathcal{J} the set $\{4, 5, 6\}$. The derivatives of the I_i with respect to \mathbf{F} and \mathbf{B}_l are given in Appendix A in component form. Explicitly we calculate the expressions for $\boldsymbol{\tau}$ for an incompressible material and \mathbf{H} as

$$\boldsymbol{\tau} = -p \mathbf{I} + 2\Omega_1 \mathbf{b} + 2\Omega_2 (I_1 \mathbf{b} - \mathbf{b}^2) + 2\Omega_5 \mathbf{B} \otimes \mathbf{B} + 2\Omega_6 (\mathbf{B} \otimes \mathbf{b} \mathbf{B} + \mathbf{b} \mathbf{B} \otimes \mathbf{B}), \tag{4.7}$$

and

$$\mathbf{H} = 2(\Omega_4 \mathbf{b}^{-1} \mathbf{B} + \Omega_5 \mathbf{B} + \Omega_6 \mathbf{b} \mathbf{B}), \tag{4.8}$$

where $I_3 \equiv 1$ and we recall that $\mathbf{b} = \mathbf{F} \mathbf{F}^T$ is the left Cauchy–Green tensor.

4.1. Magnetoelastic moduli tensors

By taking the increments of (4.1) we obtain the linearized equations

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \mathcal{C}\dot{\mathbf{B}}_l, \tag{4.9a}$$

$$\dot{\mathbf{H}}_l = \mathcal{C}^T\dot{\mathbf{F}} + \mathbf{K}\dot{\mathbf{B}}_l, \tag{4.9b}$$

where the magnetoelastic “moduli” tensors are defined by

$$\mathcal{A} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \mathcal{C} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{B}_l}, \quad \mathcal{C}^T = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{F}}, \quad \mathbf{K} = \frac{\partial^2\Omega}{\partial\mathbf{B}_l\partial\mathbf{B}_l}, \tag{4.10}$$

the products in (4.9a) and (4.9b) are defined by

$$\begin{aligned} (\mathcal{A}\dot{\mathbf{F}})_{\alpha i} &= \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, & (\mathcal{C}\dot{\mathbf{B}}_l)_{\alpha i} &= \mathcal{C}_{\alpha i|\beta}\dot{B}_{l\beta}, \\ (\mathcal{C}^T\dot{\mathbf{F}})_{\beta} &= \mathcal{C}_{\beta|\alpha i}\dot{F}_{i\alpha}, & (\mathbf{K}\dot{\mathbf{B}}_l)_{\alpha} &= \mathbf{K}_{\alpha\beta}\dot{B}_{l\beta}, \end{aligned} \tag{4.11}$$

and we note the symmetries

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i}, \quad \mathcal{C}_{\alpha i|\beta} = \mathcal{C}_{\beta|\alpha i}, \quad \mathbf{K}_{\alpha\beta} = \mathbf{K}_{\beta\alpha}, \tag{4.12}$$

which reflect the commutativity of the partial derivatives. The vertical bar between the indices on \mathcal{C} is a separator used to distinguish the single subscript from the pair of subscripts that always go together. Here and henceforth we use only Cartesian components.

For an incompressible material, (4.9a) is replaced by

$$\dot{\mathbf{T}} = \mathcal{A}\dot{\mathbf{F}} + \mathcal{C}\dot{\mathbf{B}}_l - p\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{4.13}$$

and subject to $\det \mathbf{F} = 1$, (4.9b) is unchanged.

On updating (pushing forward in Eulerian form) the incremented constitutive equations (4.9a), (4.9b) and (4.13), we obtain

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0\mathbf{L} + \mathcal{C}_0\dot{\mathbf{B}}_{l0}, \tag{4.14a}$$

$$\dot{\mathbf{H}}_{l0} = \mathcal{C}_0^T\mathbf{L} + \mathbf{K}_0\dot{\mathbf{B}}_{l0}. \tag{4.14b}$$

and

$$\dot{\mathbf{T}}_0 = \mathcal{A}_0\mathbf{L} + \mathcal{C}_0\dot{\mathbf{B}}_{l0} - p\dot{\mathbf{I}} + p\mathbf{L}, \tag{4.15}$$

respectively, where $\mathcal{A}_0, \mathcal{C}_0$ and \mathbf{K}_0 are defined in component form by

$$\mathcal{A}_{0piqj} = \mathcal{A}_{0qjpi} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\alpha i\beta j} = J^{-1}F_{p\alpha}F_{q\beta}\mathcal{A}_{\beta j\alpha i}, \tag{4.16}$$

$$\mathcal{C}_{0ij|k} = \mathcal{C}_{0k|ij} = F_{i\alpha}F_{\beta k}^{-1}\mathcal{C}_{\alpha j|\beta} = F_{i\alpha}F_{\beta k}^{-1}\mathcal{C}_{\beta|\alpha j}, \tag{4.17}$$

$$\mathbf{K}_{0ij} = \mathbf{K}_{0ji} = JF_{\alpha i}^{-1}F_{\beta j}^{-1}\mathbf{K}_{\alpha\beta}, \tag{4.18}$$

which apply for an incompressible material with $J = 1$. Explicit formulas for these components for an isotropic magnetoelastic material referred to the principal axes of the left Cauchy–Green tensor \mathbf{b} are given in Appendix B.

On substituting (4.14a) and (4.15) into (3.11) in turn, we obtain

$$\mathcal{A}_0 \mathbf{L} + \mathbf{L} \boldsymbol{\tau} = (\mathcal{A}_0 \mathbf{L})^T + \boldsymbol{\tau} \mathbf{L}^T, \quad \mathcal{C}_0 \dot{\mathbf{B}}_{l0} = (\mathcal{C}_0 \dot{\mathbf{B}}_{l0})^T \tag{4.19}$$

and

$$\mathcal{A}_0 \mathbf{L} + p \mathbf{L} + \mathbf{L} \boldsymbol{\tau} = (\mathcal{A}_0 \mathbf{L})^T + p \mathbf{L}^T + \boldsymbol{\tau} \mathbf{L}^T, \quad \mathcal{C}_0 \dot{\mathbf{B}}_{l0} = (\mathcal{C}_0 \dot{\mathbf{B}}_{l0})^T, \tag{4.20}$$

respectively, from which we deduce the symmetries

$$\mathcal{A}_{0ipqj} + \delta_{ij}(\tau_{pq} + p\delta_{pq}) = \mathcal{A}_{0piqj} + \delta_{pj}(\tau_{qi} + p\delta_{qi}), \quad \mathcal{C}_{0ij|k} = \mathcal{C}_{0ji|k}, \tag{4.21}$$

which are additional to (4.12), with $p = 0$ in the case of an unconstrained material.

Henceforth we restrict our attention to incompressible materials. We now use the constitutive Eqs. (4.15) and (4.14b) together with (3.15) to arrive at the governing equations

$$\text{curl}(\mathcal{C}_0^T \mathbf{L} + \mathbf{K}_0 \dot{\mathbf{B}}_{l0}) = \mathbf{0}, \tag{4.22a}$$

$$\text{div} \dot{\mathbf{B}}_{l0} = \mathbf{0}, \tag{4.22b}$$

$$\text{div} \mathbf{u} = 0, \tag{4.22c}$$

$$\text{div}(\mathcal{A}_0 \mathbf{L} + \mathcal{C}_0 \dot{\mathbf{B}}_{l0}) - \text{grad} \dot{p} + \mathbf{L}^T \text{grad} p = \rho \mathbf{u}_{,tt}. \tag{4.23}$$

Let us now assume that the underlying configuration is homogeneous, so that p , \mathcal{A}_0 , \mathcal{C}_0 and \mathbf{K}_0 are uniform. Then, in component form, Eqs. (4.22a)–(4.22c) and (4.23) become

$$\epsilon_{ijk}(\mathcal{C}_{0pq|k} u_{p,qj} + \mathbf{K}_{0kp} \dot{B}_{l0p,j}) = 0, \tag{4.24a}$$

$$\dot{B}_{l0i,i} = 0, \tag{4.24b}$$

$$u_{i,i} = 0, \tag{4.24c}$$

$$\mathcal{A}_{0piqj} u_{j,pq} + \mathcal{C}_{0pi|q} \dot{B}_{l0q,p} - \dot{p}_{,i} = \rho u_{i,tt}. \tag{4.25}$$

5. Homogeneous Plane Waves

We now consider infinitesimal homogenous plane waves propagating with speed v in the direction of unit vector \mathbf{n} in the form of

$$\mathbf{u} = \mathbf{m} f(\mathbf{n} \cdot \mathbf{x} - vt), \quad \dot{\mathbf{B}}_{l0} = \mathbf{q} g(\mathbf{n} \cdot \mathbf{x} - vt), \quad \dot{p} = P(\mathbf{n} \cdot \mathbf{x} - vt), \tag{5.1}$$

where \mathbf{m} and \mathbf{q} are constant (polarization) unit vectors in the directions of the incremental displacement and magnetic induction, respectively, and f , g and P are appropriately regular functions of the argument $\mathbf{n} \cdot \mathbf{x} - vt$. Substituting these

expressions into Eqs. (4.24a)–(4.24c) and (4.25), we obtain

$$\mathbf{n} \times \{\mathbf{R}(\mathbf{n})^T \mathbf{m} f'' + \mathbf{K}_0 \mathbf{q} g'\} = \mathbf{0}, \tag{5.2a}$$

$$\mathbf{q} \cdot \mathbf{n} = 0, \tag{5.2b}$$

$$\mathbf{m} \cdot \mathbf{n} = 0, \tag{5.2c}$$

$$\mathbf{Q}(\mathbf{n}) \mathbf{m} f'' + \mathbf{R}(\mathbf{n}) \mathbf{q} g' - P' \mathbf{n} = \rho v^2 \mathbf{m} f'', \tag{5.3}$$

where $\mathbf{Q}(\mathbf{n})$, the *acoustic tensor*, and $\mathbf{R}(\mathbf{n})$, the *magnetoacoustic tensor*, are given by

$$[\mathbf{Q}(\mathbf{n})]_{ij} = \mathcal{A}_{0piqj} n_p n_q, \quad [\mathbf{R}(\mathbf{n})]_{ij} = \mathcal{C}_{0ip|j} n_p, \tag{5.4}$$

and a prime signifies differentiation with respect to the argument $\mathbf{n} \cdot \mathbf{x} - vt$. Note that $\mathbf{Q}(\mathbf{n})$ is symmetric but in general $\mathbf{R}(\mathbf{n})$ is not.

Let $\hat{\mathbf{I}}(\mathbf{n}) = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ denote the symmetric projection tensor onto the plane with normal \mathbf{n} . Then, following Destrade and Ogden [2011b], we define the notations

$$\hat{\mathbf{Q}}(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n}) \mathbf{Q}(\mathbf{n}) \hat{\mathbf{I}}(\mathbf{n}), \quad \hat{\mathbf{R}}(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n}) \mathbf{R}(\mathbf{n}) \hat{\mathbf{I}}(\mathbf{n}), \quad \hat{\mathbf{K}}_0(\mathbf{n}) = \hat{\mathbf{I}}(\mathbf{n}) \mathbf{K}_0(\mathbf{n}) \hat{\mathbf{I}}(\mathbf{n}), \tag{5.5}$$

which are the projections of $\mathbf{Q}(\mathbf{n})$, $\mathbf{R}(\mathbf{n})$ and $\mathbf{K}_0(\mathbf{n})$, respectively, onto the plane normal to \mathbf{n} .

Using (5.2c) we obtain from (5.3)

$$P' = [\mathbf{Q}(\mathbf{n}) \mathbf{m}] \cdot \mathbf{n} f'' + [\mathbf{R}(\mathbf{n}) \mathbf{q}] \cdot \mathbf{n} g', \tag{5.6}$$

and substitution of this back into (5.3) enables the latter to be written as

$$\hat{\mathbf{Q}}(\mathbf{n}) \mathbf{m} f'' + \hat{\mathbf{R}}(\mathbf{n}) \mathbf{q} g' = \rho v^2 \mathbf{m} f''. \tag{5.7}$$

Similarly, from (5.2a) we deduce that

$$\mathbf{R}(\mathbf{n})^T \mathbf{m} f'' + \mathbf{K}_0 \mathbf{q} g' = \{[\mathbf{R}(\mathbf{n})^T \mathbf{m}] \cdot \mathbf{n} f'' + [\mathbf{K}_0 \mathbf{q}] \cdot \mathbf{n} g'\} \mathbf{n}, \tag{5.8}$$

which can be written more compactly as

$$\hat{\mathbf{R}}(\mathbf{n})^T \mathbf{m} f'' + \hat{\mathbf{K}}_0 \mathbf{q} g' = \mathbf{0}. \tag{5.9}$$

As in Destrade and Ogden [2011b], we assume that $\hat{\mathbf{K}}_0$ is nonsingular as an operator restricted to the plane normal to \mathbf{n} and also positive definite in view of its interpretation as the inverse of the incremental permeability tensor. We then obtain $\mathbf{q} g' = -\hat{\mathbf{K}}_0^{-1} \hat{\mathbf{R}}(\mathbf{n})^T \mathbf{m} f''$, and substitution into (5.7) and elimination of $f'' \neq 0$ yields the propagation condition for acoustic waves under the influence of a magnetic field, explicitly

$$\hat{\mathbf{P}}(\mathbf{n}) \mathbf{m} \equiv \hat{\mathbf{Q}}(\mathbf{n}) \mathbf{m} - \hat{\mathbf{R}}(\mathbf{n}) \hat{\mathbf{K}}_0^{-1} \hat{\mathbf{R}}(\mathbf{n})^T \mathbf{m} = \rho v^2 \mathbf{m}, \tag{5.10}$$

wherein the generalized acoustic (or Christoffel) tensor $\hat{\mathbf{P}}$ is defined as $\hat{\mathbf{Q}}(\mathbf{n}) - \hat{\mathbf{R}}(\mathbf{n}) \hat{\mathbf{K}}_0^{-1} \hat{\mathbf{R}}(\mathbf{n})^T$, which is symmetric. Equation (5.10) is a generalization of the

propagation condition for homogeneous plane waves in an incompressible elastic solid in the absence of a magnetic field. This prompts a corresponding generalization of the *strong ellipticity condition* in the form

$$\mathbf{m} \cdot [\hat{\mathbf{P}}(\mathbf{n})\mathbf{m}] > 0, \tag{5.11}$$

for all unit vectors \mathbf{m} and \mathbf{n} such that $\mathbf{m} \cdot \mathbf{n} = 0$, as given in Destrade and Ogden [2011b]. This guarantees that homogeneous plane waves have real wave speeds. In component form, which will be useful later, the generalized strong ellipticity inequality (5.11) can be written as

$$(\mathcal{A}_{0piqj} - \mathcal{C}_{0ip|k} \hat{K}_{0kl}^{-1} \mathcal{C}_{0jq|l}) m_i m_j n_p n_q > 0. \tag{5.12}$$

6. Two-Dimensional Specialization

Let the initial deformation of the material be given by the pure homogeneous strain

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{6.1}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are uniform. The component matrix $[\mathbf{F}]$ of the deformation gradient is then $[\mathbf{F}] = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. We also assume that the initial (uniform) magnetic induction has components $(B_1, B_2, 0)$ in the material and $(B_1^*, B_2^*, 0)$ outside.

We now study two-dimensional (2D) motions in the $(1, 2)$ plane and seek solutions depending only on the in-plane variables x_1 and x_2 such that $u_3 = \dot{B}_{l03} = \dot{B}_{l03}^* = 0$. The third component of the equation of motion (4.25) and the first two components of (4.24a) are then satisfied trivially, and the remaining equations are

$$\begin{aligned} &\mathcal{A}_{01111}u_{1,11} + 2\mathcal{A}_{01121}u_{1,12} + \mathcal{A}_{02121}u_{1,22} + \mathcal{A}_{01112}u_{2,11} \\ &\quad + (\mathcal{A}_{01122} + \mathcal{A}_{01221})u_{2,12} + \mathcal{A}_{02122}u_{2,22} + \mathcal{C}_{011|1}\dot{B}_{l01,1} \\ &\quad + \mathcal{C}_{021|1}\dot{B}_{l01,2} + \mathcal{C}_{011|2}\dot{B}_{l02,1} + \mathcal{C}_{021|2}\dot{B}_{l02,2} - \dot{p}_{,1} = \rho u_{1,tt}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} &\mathcal{A}_{01211}u_{1,11} + (\mathcal{A}_{01221} + \mathcal{A}_{01122})u_{1,12} + \mathcal{A}_{02221}u_{1,22} \\ &\quad + \mathcal{A}_{01212}u_{2,11} + 2\mathcal{A}_{01222}u_{2,12} + \mathcal{A}_{02222}u_{2,22} + \mathcal{C}_{012|1}\dot{B}_{l01,1} \\ &\quad + \mathcal{C}_{022|1}\dot{B}_{l01,2} + \mathcal{C}_{012|2}\dot{B}_{l02,1} + \mathcal{C}_{022|2}\dot{B}_{l02,2} - \dot{p}_{,2} = \rho u_{2,tt}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\mathcal{C}_{011|2}u_{1,11} + (\mathcal{C}_{021|2} - \mathcal{C}_{011|1})u_{1,12} - \mathcal{C}_{021|1}u_{1,22} \\ &\quad + \mathcal{C}_{012|2}u_{2,11} + (\mathcal{C}_{022|2} - \mathcal{C}_{012|1})u_{2,12} - \mathcal{C}_{022|1}u_{2,22} \\ &\quad + \mathcal{K}_{012}\dot{B}_{l01,1} - \mathcal{K}_{011}\dot{B}_{l01,2} + \mathcal{K}_{022}\dot{B}_{l02,1} - \mathcal{K}_{012}\dot{B}_{l02,2} = 0. \end{aligned} \tag{6.4}$$

Elimination of \dot{p} from (6.2) and (6.3) by cross differentiation and subtraction yields

$$\begin{aligned} &\mathcal{A}_{01211}u_{1,111} + (\mathcal{A}_{01221} + \mathcal{A}_{01122} - \mathcal{A}_{01111})u_{1,112} + (\mathcal{A}_{02221} - 2\mathcal{A}_{01121})u_{1,122} \\ &\quad - \mathcal{A}_{02121}u_{1,222} + \mathcal{A}_{01212}u_{2,111} + (2\mathcal{A}_{01222} - \mathcal{A}_{01112})u_{2,112} \end{aligned}$$

$$\begin{aligned}
 & -(\mathcal{A}_{01122} + \mathcal{A}_{01221} - \mathcal{A}_{02222})u_{2,122} - \mathcal{A}_{02122}u_{2,222} + \mathcal{C}_{012|1}\dot{B}_{l01,11} \\
 & + (\mathcal{C}_{022|1} - \mathcal{C}_{011|1})\dot{B}_{l01,12} - \mathcal{C}_{021|1}\dot{B}_{l01,22} + \mathcal{C}_{012|2}\dot{B}_{l02,11} \\
 & + (\mathcal{C}_{022|2} - \mathcal{C}_{011|2})\dot{B}_{l02,12} - \mathcal{C}_{021|2}\dot{B}_{l02,22} = \rho(u_{2,1} - u_{1,2}),_{tt}. \tag{6.5}
 \end{aligned}$$

The corresponding equations in (3.17) outside the material may be written as

$$\dot{B}_{1,1}^* + \dot{B}_{2,2}^* = 0, \tag{6.6a}$$

$$\dot{B}_{2,1}^* - \dot{B}_{1,2}^* = 0. \tag{6.6b}$$

Since $\dot{\mathbf{B}}_{l0}$ and \mathbf{u} satisfy the Eqs. (4.22b) and (4.22c) and $\dot{\mathbf{B}}^*$ satisfies Eq. (6.6a), we can define potentials ψ , ϕ and ψ^* such that

$$\dot{B}_{l01} = \psi_{,2}, \quad \dot{B}_{l02} = -\psi_{,1}, \quad u_1 = \phi_{,2}, \quad u_2 = -\phi_{,1}, \quad \dot{B}_1^* = \psi_{,2}^*, \quad \dot{B}_2^* = -\psi_{,1}^*. \tag{6.7}$$

Substituting these expressions into the governing Eqs. (6.5), (6.4) and (6.6b), we obtain the two coupled equations

$$\begin{aligned}
 & \alpha\phi_{,1111} + 2\delta\phi_{,1112} + 2\beta\phi_{,1122} + 2\varepsilon\phi_{,1222} + \gamma\phi_{,2222} + a\psi_{,111} + b\psi_{,112} \\
 & + c\psi_{,122} + d\psi_{,222} = \rho(\phi_{,11} + \phi_{,22}),_{tt}, \tag{6.8}
 \end{aligned}$$

$$a\phi_{,111} + b\phi_{,112} + c\phi_{,122} + d\phi_{,222} + \mathbf{K}_{011}\psi_{,22} + \mathbf{K}_{022}\psi_{,11} - 2\mathbf{K}_{012}\psi_{,12} = 0, \tag{6.9}$$

for ϕ and ψ in the material, where, for compactness of representation, we have introduced the notations

$$\alpha = \mathcal{A}_{01212}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}, \quad \gamma = \mathcal{A}_{02121}, \tag{6.10}$$

$$\delta = \mathcal{A}_{01222} - \mathcal{A}_{01211}, \quad \varepsilon = \mathcal{A}_{01121} - \mathcal{A}_{02221}, \quad a = \mathcal{C}_{012|2}, \tag{6.11}$$

$$b = \mathcal{C}_{022|2} - \mathcal{C}_{011|2} - \mathcal{C}_{012|1}, \quad c = \mathcal{C}_{011|1} - \mathcal{C}_{022|1} - \mathcal{C}_{021|2}, \quad d = \mathcal{C}_{021|1}. \tag{6.12}$$

Outside the material we have the single equation

$$\psi_{,11}^* + \psi_{,22}^* = 0. \tag{6.13}$$

When there is no time dependence and $B_1 = 0$, Eqs. (6.8) and (6.9) reduce to equations given in Sec. 5.2 of Otténio *et al.* [2008], but partly in different notation.

7. Surface Waves

In this section, we consider two separate cases: first, $B_1 = 0$ with $B_2 \neq 0$; and second, $B_1 \neq 0$ with $B_2 = 0$. The material forms a half-space $X_2 < 0$ in the reference configuration, with unit outward normal \mathbf{N} to its boundary $X_2 = 0$ having components $(0, 1, 0)$. Under the deformation (6.1), the material occupies the half-space $x_2 < 0$ in the deformed configuration and the unit outward normal \mathbf{n} to its boundary $x_2 = 0$ has components $(0, 1, 0)$.

7.1. Magnetic induction components (0, B₂, 0)

In this first example, we take the initial magnetic induction to be perpendicular to the surface of the half-space so that the components of **B** are (0, B₂, 0). The boundary condition **B** · **n** = **B**^{*} · **n** applied to x₂ = 0 then gives B₂^{*} = B₂. It follows from (3.21) and (3.22) that the matrix representations of **τ**^{*} and **ṡ**^{*} are, respectively,

$$[\boldsymbol{\tau}^*] = \frac{B_2}{2\mu_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_2}{\mu_0} \begin{bmatrix} -\dot{B}_2^* & \dot{B}_1^* & 0 \\ \dot{B}_1^* & \dot{B}_2^* & 0 \\ 0 & 0 & -\dot{B}_2^* \end{bmatrix}. \tag{7.1}$$

7.1.1. *Incremental equations and boundary conditions*

For the given values of **F** and **B**, many of the components of the moduli listed in Appendix B vanish, and Eqs. (6.8) and (6.9) simplify to

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} + b\psi_{,112} + d\psi_{,222} = \rho(\phi_{,11} + \phi_{,22})_{,tt}, \tag{7.2}$$

$$b\phi_{,112} + d\phi_{,222} + K_{011}\psi_{,22} + K_{022}\psi_{,11} = 0. \tag{7.3}$$

Using the values of **τ**^{*} and **ṡ**^{*} from (7.1) and assuming there is no incremental mechanical traction on x₂ = 0 the components of the incremental traction are obtained from (3.25) with **T**₀^T**N** = **t**_{M0} as

$$\dot{T}_{021} - \frac{B_2}{\mu_0}\dot{B}_1^* - \frac{B_2^2}{2\mu_0}u_{2,1} = 0, \quad \dot{T}_{022} - \frac{B_2}{\mu_0}\dot{B}_2^* + \frac{B_2^2}{2\mu_0}u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{7.4}$$

with $\dot{T}_{023} = 0$ satisfied identically. From (3.19) and (3.20) we obtain

$$\dot{B}_{l02} - \dot{B}_2^* + B_2u_{2,2} = 0, \tag{7.5a}$$

$$\dot{H}_{l01} - \frac{B_2}{\mu_0}u_{2,1} - \dot{H}_1^* = 0 \quad \text{on } x_2 = 0. \tag{7.5b}$$

By substituting the updated incremented constitutive equations (4.15) and (4.14b), appropriately specialized, into the incremental boundary conditions (7.4) and (7.5a) and making use of the connection

$$\mathcal{A}_{01221} + \tau_{22} + p = \mathcal{A}_{02121}, \tag{7.6}$$

which comes from (4.21), we obtain

$$\left(\mathcal{A}_{02121} - \tau_{22} - \frac{B_2^2}{2\mu_0}\right)u_{2,1} + \mathcal{A}_{02121}u_{1,2} + \mathcal{C}_{021|1}\dot{B}_{l01} - \frac{B_2}{\mu_0}\dot{B}_1^* = 0, \tag{7.7}$$

$$\mathcal{A}_{01122}u_{1,1} + \left(\mathcal{A}_{02222} + p + \frac{B_2^2}{2\mu_0}\right)u_{2,2} + \mathcal{C}_{022|2}\dot{B}_{l02} - \dot{p} - \frac{B_2}{\mu_0}\dot{B}_2^* = 0, \tag{7.8}$$

$$\mathcal{C}_{012|1}u_{2,1} + \mathcal{C}_{021|1}u_{1,2} + K_{011}\dot{B}_{l01} - \frac{B_2}{\mu_0}u_{2,1} - \frac{1}{\mu_0}\dot{B}_1^* = 0, \tag{7.9}$$

each holding on x₂ = 0.

Next, we differentiate (7.8) with respect to x_1 and make use of (6.2) to eliminate $\dot{p}_{,1}$. We then introduce the potentials ϕ, ψ and ψ^* into the resulting equation and Eqs. (7.5a), (7.5b), (7.7) and (7.9) and use the notations (6.10)–(6.12). We also note that if there is no mechanical traction applied on the boundary $x_2 = 0$ in the underlying configuration then the normal stress τ_{22} in the material must balance the Maxwell stress τ_{22}^* on $x_2 = 0$, which gives

$$\tau_{22} = \tau_{22}^* = \frac{B_2^2}{2\mu_0}. \tag{7.10}$$

The boundary conditions can then be written as

$$(\gamma - 2\tau_{22}^*)\phi_{,11} - \gamma\phi_{,22} - d\psi_{,2} + \frac{B_2}{\mu_0}\psi_{,2}^* = 0, \tag{7.11}$$

$$(2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} + (b + d)\psi_{,11} + d\psi_{,22} - \frac{B_2}{\mu_0}\psi_{,11}^* - \rho\phi_{,2tt} = 0, \tag{7.12}$$

$$B_2\phi_{,12} + \psi_{,1} - \psi_{,1}^* = 0, \tag{7.13}$$

$$d(\phi_{,11} - \phi_{,22}) - K_{011}\psi_{,2} - \frac{B_2}{\mu_0}\phi_{,11} + \frac{1}{\mu_0}\psi_{,2}^* = 0, \tag{7.14}$$

which apply on $x_2 = 0$.

Hence, the problem is reduced to solving the governing Eqs. (7.2) and (7.3) in $x_2 < 0$ and (6.13) in $x_2 > 0$ and applying the boundary conditions (7.11)–(7.14) on $x_2 = 0$ and appropriate decay behavior as $x_2 \rightarrow \pm\infty$.

7.1.2. Surface wave propagation

We now study 2D surface waves propagating in the x_1 direction with the increments having nonzero components lying in the (1, 2) plane. We consider harmonic solutions of the form

$$\phi = P \exp(skx_2 + ikx_1 - i\omega t), \tag{7.15}$$

$$\psi = kQ \exp(skx_2 + ikx_1 - i\omega t) \quad \text{in } x_2 < 0,$$

$$\psi^* = kR \exp(s^*kx_2 + ikx_1 - i\omega t) \quad \text{in } x_2 > 0, \tag{7.16}$$

where P, Q, R are constants, k is the wave number and ω the angular frequency, and s and s^* are to be determined subject to the requirements $\text{Re}(s) > 0$ and $\text{Re}(s^*) < 0$ needed for decay of the surface wave amplitude away from the boundary. Substituting these solutions into the governing equations (7.2), (7.3) and (6.13), we obtain

$$[\alpha - 2\beta s^2 + \gamma s^4 + \rho v^2(s^2 - 1)]P + (ds^2 - b)sQ = 0, \tag{7.17}$$

$$(ds^2 - b)sP + (K_{011}s^2 - K_{022})Q = 0, \tag{7.18}$$

and $s^{*2} = 1$, where the wave speed is $v = \omega/k$. For the solution ψ^* to decay as $x_2 \rightarrow \infty$, we necessarily take $s^* = -1$. For nontrivial solutions for P and Q from

(7.17) and (7.18), we set the determinant of coefficients to be zero and obtain a cubic equation for s^2 , namely

$$(\gamma K_{011} - d^2)s^6 + [K_{011}(\rho v^2 - 2\beta) - \gamma K_{022} + 2bd]s^4 + [K_{022}(2\beta - \rho v^2) + K_{011}(\alpha - \rho v^2) - b^2]s^2 + (\rho v^2 - \alpha)K_{022} = 0. \tag{7.19}$$

We denote by s_1, s_2, s_3 the three solutions satisfying the requirement $\text{Re}(s) > 0$. The general solutions of the equations that satisfy the decay conditions are now given by

$$\phi = (P_1 e^{s_1 kx_2} + P_2 e^{s_2 kx_2} + P_3 e^{s_3 kx_2}) e^{i(kx_1 - \omega t)}, \tag{7.20}$$

$$\psi = k(Q_1 e^{s_1 kx_2} + Q_2 e^{s_2 kx_2} + Q_3 e^{s_3 kx_2}) e^{i(kx_1 - \omega t)}, \tag{7.21}$$

$$\psi^* = k \text{Re} e^{-kx_2 + i(kx_1 - \omega t)}. \tag{7.22}$$

For each i, Q_i is related to P_i by Eq. (7.18), which we rewrite here as

$$Q_i = \frac{(b - ds_i^2)s_i}{K_{011}s_i^2 - K_{022}} P_i, \quad i = 1, 2, 3. \tag{7.23}$$

Next, we substitute the general solutions (7.20)–(7.22) into the boundary conditions (7.11)–(7.14) to obtain

$$(\gamma - 2\tau_{22}^*) \sum_j P_j + \gamma \sum_j s_j^2 P_j + d \sum_j s_j Q_j + \frac{B_2}{\mu_0} R = 0, \tag{7.24}$$

$$(2\beta + \gamma - \rho v^2) \sum_j s_j P_j - \gamma \sum_j s_j^3 P_j + (b + d) \sum_j Q_j - d \sum_j s_j^2 Q_j - \frac{B_2}{\mu_0} R = 0, \tag{7.25}$$

$$B_2 \sum_j s_j P_j + \sum_j Q_j - R = 0, \tag{7.26}$$

$$d \sum_j (s_j^2 + 1) P_j + K_{011} \sum_j s_j Q_j - \frac{B_2}{\mu_0} \sum_j P_j + \frac{1}{\mu_0} R = 0, \tag{7.27}$$

where \sum_j indicates summation over j from 1 to 3.

We now have seven linear equations in $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and R , and for a nontrivial solution the determinant of coefficients must vanish. The result is the secular equation relating the wave speed v to the initial deformation, the material properties and the initial magnetic induction B_2 , and we note that, by (7.10), the stress τ_{22}^* depends on B_2 .

7.1.3. *Pure elastic case*

Here we take the magnetic field to vanish in order to reduce our results to known results in the purely elastic case. For this purpose we set $\mathbf{C} = \mathbf{0}, Q_i = 0, i = 1, 2, 3$

and $R = 0$. Equation (7.19) reduces to a quadratic for s^2 , namely

$$\gamma s^4 - (2\beta - \rho v^2)s^2 + \alpha - \rho v^2 = 0, \tag{7.28}$$

from which we deduce that the solutions s_1^2 and s_2^2 satisfy

$$\gamma(s_1^2 + s_2^2) = 2\beta - \rho v^2, \quad \gamma s_1^2 s_2^2 = \alpha - \rho v^2. \tag{7.29}$$

For a surface wave we take s_1 and s_2 to be the solutions satisfying $\text{Re}(s) > 0$, and, as discussed in Dowaiikh and Ogden [1990], we require $\gamma > 0$ and $\rho v^2 \leq \alpha$.

The boundary conditions (7.11)–(7.14) reduce to the two equations

$$(\gamma - \tau_{22})\phi_{,11} - \gamma\phi_{,22} = 0, \quad (2\beta + \gamma - \tau_{22})\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} = 0, \tag{7.30}$$

which hold on $x_2 = 0$, where, for comparison with the results of Dowaiikh and Ogden [1990], we have assumed that there is a normal mechanical traction τ_{22} on $x_2 = 0$ in the underlying configuration. The general solution for ϕ can be rewritten as

$$\phi = (P_1 e^{s_1 k x_2} + P_2 e^{s_2 k x_2}) e^{i(kx_1 - \omega t)}. \tag{7.31}$$

Substitution into the boundary conditions then yields

$$(\gamma - \tau_{22} + \gamma s_1^2)P_1 + (\gamma - \tau_{22} + \gamma s_2^2)P_2 = 0, \tag{7.32}$$

$$(2\beta + \gamma - \tau_{22} - \rho v^2 - \gamma s_1^2)s_1 P_1 + (2\beta + \gamma - \tau_{22} - \rho v^2 - \gamma s_2^2)s_2 P_2 = 0, \tag{7.33}$$

from which, on use of (7.29), the explicit secular equation is obtained as

$$\gamma(\alpha - \rho v^2) + (2\beta + 2\gamma - 2\tau_{22} - \rho v^2)\sqrt{\gamma(\alpha - \rho v^2)} = (\gamma - \tau_{22})^2. \tag{7.34}$$

Apart from some minor differences of notation, this agrees with the formula (5.17) obtained by Dowaiikh and Ogden [1990].

7.1.4. Application to a Mooney–Rivlin magnetoelastic material

For purposes of illustration we now consider the energy function of a Mooney–Rivlin magnetoelastic material as used by Otténio *et al.* [2008]. This has the form

$$\Omega = \frac{1}{4}\mu(0)[(1+n)(I_1 - 3) + (1-n)(I_2 - 3)] + lI_4 + mI_5, \tag{7.35}$$

where $\mu(0)$ is the shear modulus of the material in the absence of magnetic fields and, to avoid a conflict of notation, we use l, m, n , respectively, in place of the $\alpha/\mu_0, \beta/\mu_0, \gamma$ used in Otténio *et al.* [2008]. Note that $l\mu_0, m\mu_0$ and n are dimensionless, with n restricted to the range $-1 \leq n \leq 1$, as for the classical Mooney–Rivlin model. For the reasons discussed in Otténio *et al.* [2008] we take l and m to be non-negative parameters.

The relevant nonzero components of the magnetoelastic tensors are easily calculated from the formulas in Appendix B as

$$\mathcal{A}_{01111} = \frac{1}{2}\mu(0)\lambda_1^2[1 + n + (1 - n)(\lambda_2^2 + \lambda_3^2)], \tag{7.36}$$

$$\mathcal{A}_{02222} = \frac{1}{2}\mu(0)\lambda_2^2[1 + n + (1 - n)(\lambda_1^2 + \lambda_3^2)] + 2mB_2^2, \tag{7.37}$$

$$\mathcal{A}_{01212} = \frac{1}{2}\mu(0)\lambda_1^2[1 + n + (1 - n)\lambda_3^2], \quad \mathcal{A}_{02121} = \lambda_2^2\lambda_1^{-2}\mathcal{A}_{01212} + 2mB_2^2, \tag{7.38}$$

$$\mathcal{A}_{01122} = -2\mathcal{A}_{01221} = \mu(0)(1 - n)\lambda_1^2\lambda_2^2, \tag{7.39}$$

$$\mathcal{C}_{022|2} = 2\mathcal{C}_{012|1} = 4mB_2, \tag{7.40}$$

$$\mathbf{K}_{011} = 2(m + \lambda_1^{-2}l), \quad \mathbf{K}_{022} = 2(m + \lambda_2^{-2}l), \tag{7.41}$$

from which we deduce, using the notation defined in (6.10)–(6.12), that

$$2\beta = \alpha + \gamma, \quad b = d. \tag{7.42}$$

With these values, Eq. (7.19) factorizes in the form

$$(s^2 - 1)\{(\gamma\mathbf{K}_{011} - d^2)s^4 - [\gamma\mathbf{K}_{022} + (\alpha - \rho v^2)\mathbf{K}_{011} - d^2]s^2 + (\alpha - \rho v^2)\mathbf{K}_{022}\} = 0. \tag{7.43}$$

Let the solutions with positive real part be denoted $s_1 (= 1)$, s_2 and s_3 . Then,

$$s_2^2 + s_3^2 = \frac{\gamma\mathbf{K}_{022} + (\alpha - \rho v^2)\mathbf{K}_{011} - d^2}{\gamma\mathbf{K}_{011} - d^2}, \tag{7.44a}$$

$$s_2^2 s_3^2 = \frac{(\alpha - \rho v^2)\mathbf{K}_{022}}{\gamma\mathbf{K}_{011} - d^2}. \tag{7.44b}$$

Note that when $v = 0$ the bi-quadratic in (7.43) factorizes easily to give the equation

$$(s^2 - \lambda^4)[(\mu_0\mathbf{K}_{011} + 4lmB_2^2)s^2 - \mu_0\mathbf{K}_{022}] = 0, \tag{7.45}$$

as shown by Otténio *et al.* [2008], although there is a slight error in their Eq. (112), wherein their α and β should be replaced by 2α and 2β , respectively. This has only minor repercussions for their subsequent results. We also note in passing that for $v \neq 0$, in the special case $\lambda = 1$, the bi-quadratic factorizes as $(s^2 - 1)[(\gamma\mathbf{K}_{011} - d^2)s^2 - \mathbf{K}_{011}(\alpha - \rho v^2)]$.

Now, by specializing the generalized strong ellipticity condition (5.12) to the present constitutive model and setting $n_1 = 1, n_2 = 0, m_1 = 0, m_2 = 1$ we obtain $\gamma\mathbf{K}_{011} - d^2 > 0$. Then, following the same argument as used in the purely elastic case, we require $s_2^2 s_3^2 \geq 0$ and we therefore conclude from (7.44b) that

$$\rho v^2 \leq \alpha. \tag{7.46}$$

For the considered model, this upper bound is identical to that in the purely elastic case and hence independent of the magnetic field.

We now use $s_1 = 1$ and the expressions (7.44a) and (7.44b) in the boundary conditions (7.24)–(7.27) and set the determinant of coefficients to zero to obtain the secular equation. The resulting equation is too lengthy to reproduce here, and we obtain the solutions numerically. For this purpose, we use the standard value $1.257 \times 10^{-6} \text{ NA}^{-2}$ of μ_0 together with the value $2.6 \times 10^5 \text{ Nm}^{-2}$ of $\mu(0)$ that was adopted by Otténio *et al.* [2008] based on data for an elastomer filled with 10% by volume of iron particles from Jolly *et al.* [1996]. We also use a series of values of l and m consistent with the values of the magnetoelastic coupling constants used in Otténio *et al.* [2008].

First, we consider the underlying deformation to be one of plane strain in the (1, 2) plane, and we take $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$. In this case, the results are independent of the parameter n in the Mooney–Rivlin model and the upper bound (7.46) is $\mu(0)\lambda^2$. Let $\zeta = \rho v^2/\mu(0)$. Then we plot the variation of ζ with λ for a selection of values of l and m and a range of values of B_2 in Figs. 1 and 2. We also consider a deformation for which $\lambda_1 = 1$, $\lambda_2 = \lambda$, $\lambda_3 = \lambda^{-1}$ and we use the value $n = 0.3$ in the Mooney–Rivlin model. Then, the upper bound (7.46) is $\mu(0)(0.65 + 0.35\lambda^{-2})$. Results for this case are plotted in Fig. 3 for two representative pairs of values of l and m and a range of values of B_2 .

Figures 1 and 2 relate to a plane strain deformation in which the half-space is subject to compression or extension parallel to its boundary. The result for $B_2 = 0$ corresponds to the purely elastic case and provides a point of reference. The $B_2 = 0$ curve cuts the λ axis at $\lambda = \lambda_c \simeq 0.5437$, which agrees with the classical result for the critical value of λ corresponding to loss of stability of the half-space under compression for the neo-Hookean model (for which $n = 1$); see Biot [1965] and Dowaikh

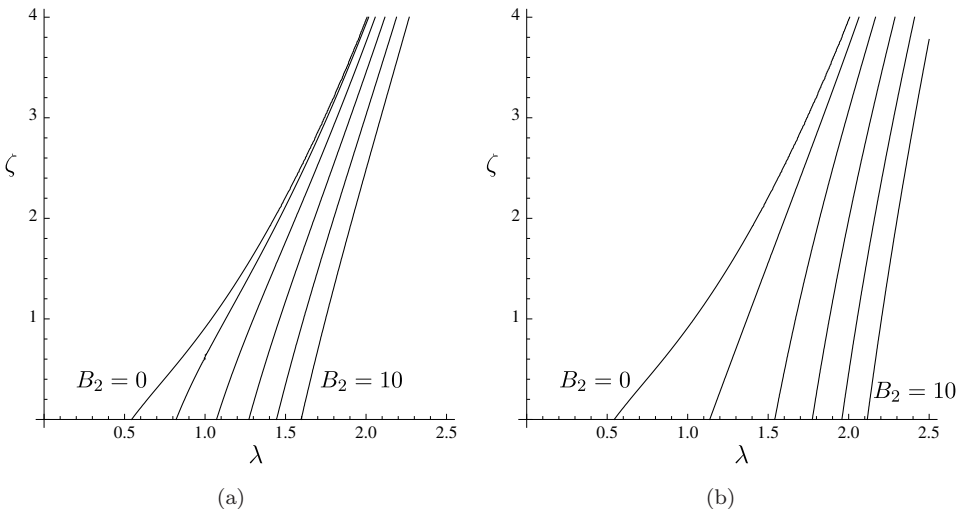


Fig. 1. Plot of $\zeta = \rho v^2/\mu(0)$ versus $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_2 = 0, 2, 4, 6, 8, 10$ T (curves reading from left to right): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 0.1, \mu_0 m = 1$.

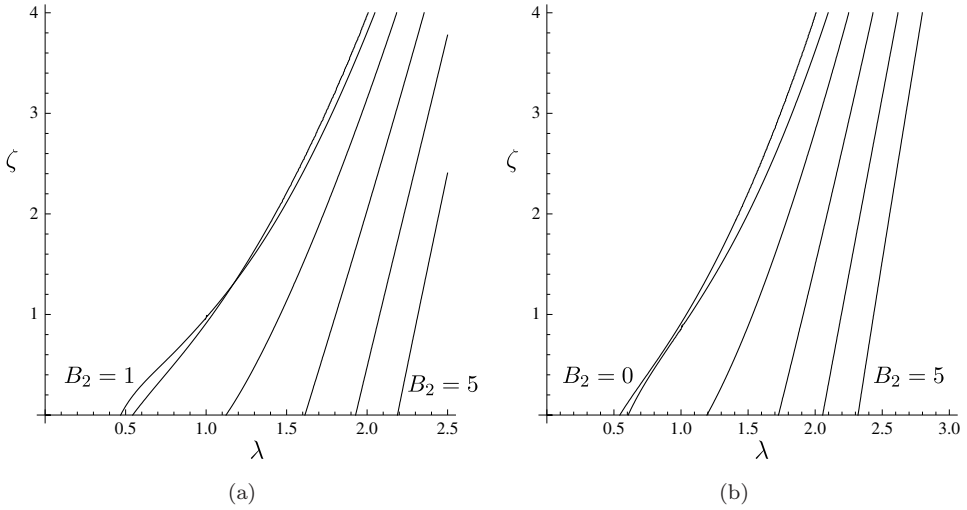


Fig. 2. Plot of $\zeta = \rho v^2/\mu(0)$ versus $\lambda_1 = \lambda$ with $\lambda_3 = 1$: (a) $\mu_0 l = 2, \mu_0 m = 0.2$ with $B_2 = 1, 0, 2, 3, 4, 5$ T; (b) $\mu_0 l = 0.2, \mu_0 m = 0.2$ with $B_2 = 0, 1, 2, 3, 4, 5$ T (curves reading from left to right in each case).

and Ogden [1990] for details. By referring to the $\zeta = 0$ axis in Figs. 1 and 2(b) it can be seen that the magnetic field destabilizes the material, i.e., instability occurs at a compression closer to the undeformed configuration where $\lambda = 1$. For each value of B_2 there is a critical value of λ beyond which a surface wave exists, and the wave speed increases with λ consistently with the upper bound (7.46). Note, in particular, that the undeformed configuration $\lambda = 1$ becomes unstable as B_2 increases. In Fig. 2(a) the situation is slightly different since for small values of B_2 the half-space is initially stabilized as B_2 increases (i.e., the critical value of λ decreases below the classical value λ_c), but then as B_2 is increased further stability is lost again. Note that the $B_2 = 0$ and $B_2 = 1$ curves cross over in this case. These results are consistent with the stability analysis of Otténio *et al.* [2008].

When there is no compression or extension parallel to $x_2 = 0$ in the sagittal plane but there is extension (or compression) normal to the boundary and a corresponding compression (or extension) normal to the sagittal plane the effect of the magnetic field is different. Figure 3 illustrates this case. Now there is instability for $\lambda > 1$, at $\lambda \simeq 3.4$ for $B_2 = 0$, and the critical value of λ decreases with increasing B_2 , i.e., the magnetic field again has a destabilizing effect. The wave speed increases as λ decreases, again consistently with the upper bound (7.46).

Figure 4 shows plots of the dimensionless squared wave speed as a function of B_2 for the undeformed configuration $\lambda = 1$ for (a) a fixed value of m and a series of values of l , and (b) a fixed value of l and a series of values of m . For $B_2 = 0$ the curves cut the ζ axis at the classical Rayleigh value ($\simeq 0.9126$). As B_2 increases then, depending on the values of the parameters l and m , the wave speed either increases

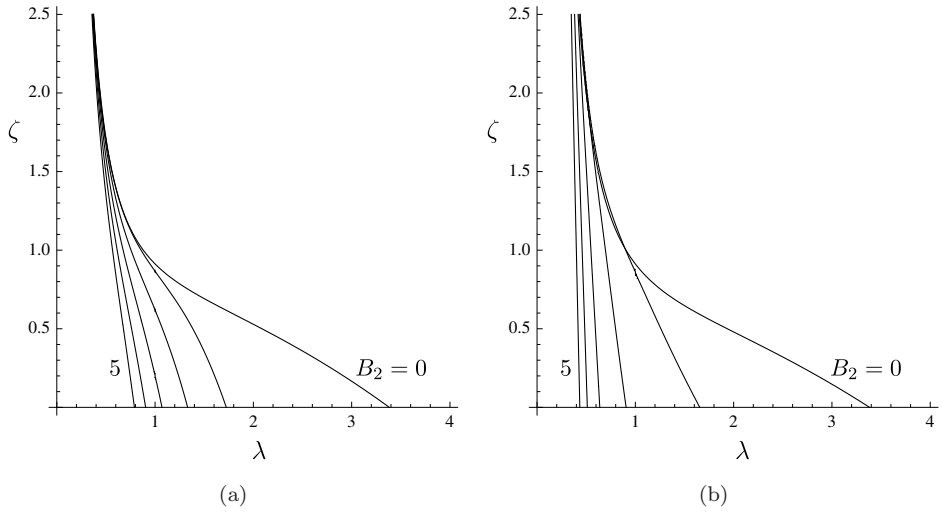


Fig. 3. Plot of $\zeta = \rho v^2 / \mu(0)$ versus $\lambda_2 = \lambda$ with $\lambda_1 = 1$ for $B_2 = 0, 1, 2, 3, 4, 5$ T (curves reading from right to left): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 0.2, \mu_0 m = 0.2$.

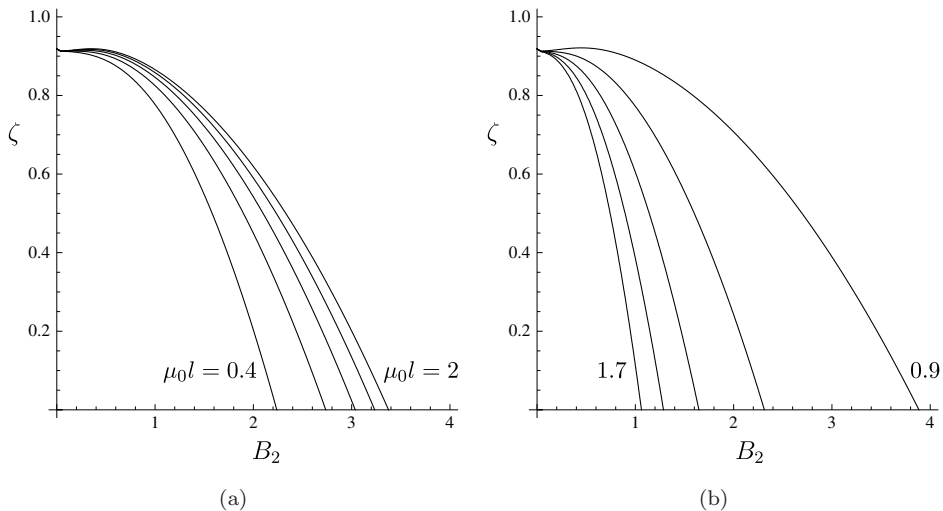


Fig. 4. Plot of $\zeta = \rho v^2 / \mu(0)$ versus B_2 with $\lambda_1 = \lambda_2 = \lambda_3 = 1$: (a) $\mu_0 l = 0.4, 0.8, 1.2, 1.6, 2$ and $\mu_0 m = 1$ (curves reading from left to right); (b) $\mu_0 l = 1$ and $\mu_0 m = 0.9, 1.1, 1.3, 1.5, 1.7$ (curves reading from right to left).

or decreases initially but in each case subsequently decreases to zero with further increase in B_2 . This emphasizes that the undeformed configuration is destabilized at a critical value of B_2 dependent on the material parameters. From Fig. 4(a), for the selected value of m , it can be seen that increasing the value of l has a stabilizing effect, while from Fig. 4(b) the reverse is true for increasing m at a fixed value of l .

7.2. Magnetic induction components $(B_1, 0, 0)$

The initial deformed configuration is considered to be the same as in Sec. 7.1, but now we take the magnetic induction \mathbf{B} to have components $(B_1, 0, 0)$. The corresponding magnetic field \mathbf{H} is given by (4.8) and has components $(H_1, 0, 0)$, with

$$H_1 = 2(\Omega_4\lambda_1^{-2} + \Omega_5 + \Omega_6\lambda_1^2)B_1, \tag{7.47}$$

which, for the model (7.35), reduces to $H_1 = 2(l\lambda_1^{-2} + m)B_1$. The magnetic boundary conditions on $x_2 = 0$ require that $H_1^* = H_1$, so that $B_1^* = \mu_0 H_1^* = 2\mu_0(l\lambda_1^{-2} + m)B_1$.

From (3.21) and (3.22), the Maxwell stress and its increment in $x_2 > 0$ are given by

$$[\boldsymbol{\tau}^*] = \frac{B_1^{*2}}{2\mu_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_1^*}{\mu_0} \begin{bmatrix} \dot{B}_1^* & \dot{B}_2^* & 0 \\ \dot{B}_2^* & -\dot{B}_1^* & 0 \\ 0 & 0 & -\dot{B}_1^* \end{bmatrix}. \tag{7.48}$$

7.2.1. Incremental equations and boundary conditions

For the present situation, Eqs. (6.8) and (6.9) reduce to

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} + a\psi_{,111} + c\psi_{,122} = \rho(\phi_{,11} + \phi_{,22})_{,tt}, \tag{7.49}$$

$$a\phi_{,111} + c\phi_{,122} + K_{022}\psi_{,11} + K_{011}\psi_{,22} = 0 \tag{7.50}$$

for $x_2 < 0$, while again (6.13) holds for $x_2 > 0$.

Using the values of $\boldsymbol{\tau}^*$ and $\dot{\boldsymbol{\tau}}^*$ from (7.48) and assuming there is no incremental mechanical traction on $x_2 = 0$ the components of the incremental traction are obtained from (3.25) with $\dot{\mathbf{T}}_0^T \mathbf{N} = \dot{\mathbf{t}}_{M0}$ as

$$\dot{T}_{021} - \frac{B_1^*}{\mu_0} \dot{B}_2^* + \frac{B_1^{*2}}{2\mu_0} u_{2,1} = 0, \quad \dot{T}_{022} + \frac{B_1^*}{\mu_0} \dot{B}_1^* - \frac{B_1^{*2}}{2\mu_0} u_{2,2} = 0 \quad \text{on } x_2 = 0, \tag{7.51}$$

with $\dot{T}_{023} = 0$ satisfied identically. From (3.19) and (3.20) we obtain

$$\dot{B}_{i02} - \dot{B}_2^* + B_1^* u_{2,1} = 0, \quad \dot{H}_{i01} - H_1^* u_{1,1} - \dot{H}_1^* = 0 \quad \text{on } x_2 = 0. \tag{7.52}$$

Next, we substitute the updated incremented constitutive Eqs. (4.14) and (4.15) into Eqs. (7.51) and (7.52) and use (7.6) and the boundary condition $\tau_{22} = \tau_{22}^*$, where $\tau_{22}^* = -B_1^{*2}/2\mu_0$, and follow the same procedure as in the previous section to eliminate \dot{p} . This yields

$$(\gamma - 2\tau_{22}^*)\phi_{,11} - \gamma\phi_{,22} + a\psi_{,1} - \frac{B_1^*}{\mu_0}\psi_{,1}^* = 0, \tag{7.53}$$

$$(2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} + c\psi_{,12} - \frac{B_1^*}{\mu_0}\psi_{,12}^* = 0, \tag{7.54}$$

$$B_1^* \phi_{,11} + \psi_{,1} - \psi_{,1}^* = 0, \tag{7.55}$$

$$\left(c + a - \frac{B_1^*}{\mu_0} \right) \phi_{,12} - \frac{1}{\mu_0} \psi_{,2}^* = 0, \tag{7.56}$$

each of which holds on $x_2 = 0$.

7.2.2. Surface waves in a Mooney–Rivlin magnetoelastic half-space

We again study surface waves as in Sec. 7.1, with solutions of the form (7.15) and (7.16). Substituting these solutions into Eqs. (7.49), (7.50) and (6.13), we obtain

$$[\gamma s^4 - (2\beta - \rho v^2)s^2 + \alpha - \rho v^2]P + i(cs^2 - a)Q = 0, \tag{7.57}$$

$$i(cs^2 - a)P + (K_{011}s^2 - K_{022})Q = 0, \tag{7.58}$$

and $s^{*2} = 1$, where the wave speed is again given by $v = \omega/k$.

For the solution ψ^* to decay as $x_2 \rightarrow \infty$, we take $s^* = -1$. For nontrivial solutions for P and Q , we set the determinant of coefficients to zero and obtain a cubic equation in s^2 :

$$\begin{aligned} & \gamma K_{011}s^6 - [K_{011}(2\beta - \rho v^2) + \gamma K_{022} - c^2]s^4 \\ & + [K_{011}(\alpha - \rho v^2) + K_{022}(2\beta - \rho v^2) - 2ac]s^2 \\ & - K_{022}(\alpha - \rho v^2) + a^2 = 0. \end{aligned} \tag{7.59}$$

For the Mooney–Rivlin magnetoelastic material given by (7.35), the nonzero components of the magnetoelastic tensors are obtained from the general formulas in Appendix B as

$$\mathcal{A}_{01111} = \frac{1}{2}\mu(0)\lambda_1^2[1 + n + (1 - n)(\lambda_2^2 + \lambda_3^2)] + 2mB_1^2, \tag{7.60}$$

$$\mathcal{A}_{02222} = \frac{1}{2}\mu(0)\lambda_2^2[1 + n + (1 - n)(\lambda_1^2 + \lambda_3^2)], \tag{7.61}$$

$$\mathcal{A}_{02121} = \frac{1}{2}\mu(0)\lambda_2^2[1 + n + (1 - n)\lambda_3^2], \quad \mathcal{A}_{01212} = \lambda_1^2\lambda_2^{-2}\mathcal{A}_{02121} + 2mB_1^2, \tag{7.62}$$

$$\mathcal{A}_{01122} = -2\mathcal{A}_{01221} = \mu(0)(1 - n)\lambda_1^2\lambda_2^2, \tag{7.63}$$

$$\mathcal{C}_{011|1} = 2\mathcal{C}_{012|2} = 4mB_1, \tag{7.64}$$

$$K_{011} = 2(m + \lambda_1^{-2}l), \quad K_{022} = 2(m + \lambda_2^{-2}l), \tag{7.65}$$

from which, using the notation defined in (6.10)–(6.12), we obtain

$$2\beta = \alpha + \gamma, \quad c = a. \tag{7.66}$$

Substitution of these values in (7.59) yields the factorization

$$(s^2 - 1)\{\gamma K_{011}s^4 - [\gamma K_{022} + (\alpha - \rho v^2)K_{011} - a^2]s^2 + (\alpha - \rho v^2)K_{022} - a^2\} = 0. \tag{7.67}$$

We note in passing that the second factor in the above can be factorized in simple form in two cases: for $v = 0$, we obtain $(s^2 - \lambda^4)(\gamma K_{011}s^2 - \gamma K_{022} - 4lm\lambda^{-2}B_1^2)$; for $\lambda = 1$, the result is $(s^2 - 1)[\gamma K_{011}s^2 + a^2 - K_{011}(\alpha - \rho v^2)]$.

Let $s_1 = 1$, and let s_2 and s_3 be the solutions of the second factor with positive real part. As in the previous section we require $s_2^2 s_3^2 \geq 0$, which, after noting that $\gamma > 0$, $K_{011} > 0$ and $K_{022} > 0$ and specializing the generalized strong ellipticity condition as in Sec. 7.1, gives

$$\rho v^2 \leq \alpha - a^2 / K_{022}, \tag{7.68}$$

the right-hand side of which is positive. As distinct from (7.46) the upper bound in (7.68) *does* in general depend on the magnetic field.

We again take the solutions for ϕ , ψ and ψ^* as (7.15) and (7.16). Substituting these into the boundary conditions (7.53)–(7.56), we obtain

$$(\gamma - 2\tau_{22}^*) \sum_j P_j + \gamma \sum_j s_j^2 P_j - ia \sum_j Q_j + i \frac{B_1^*}{\mu_0} R = 0, \tag{7.69}$$

$$(2\beta + \gamma - \rho v^2) \sum_j s_j P_j - \gamma \sum_j s_j^3 P_j - ic \sum_j s_j Q_j - i \frac{B_1^*}{\mu_0} R = 0, \tag{7.70}$$

$$B_1^* \sum_j P_j - i \sum_j Q_j + iR = 0, \tag{7.71}$$

$$\left(c + a - \frac{B_1^*}{\mu_0}\right) \sum_j s_j P_j - i \frac{1}{\mu_0} R = 0, \tag{7.72}$$

along with the connection between Q_i and P_i from (7.58):

$$Q_i = \frac{i(a - cs_i^2)}{K_{011}s_i^2 - K_{022}} P_i, \quad i = 1, 2, 3. \tag{7.73}$$

Again, \sum_j signifies summation over j from 1 to 3.

As in the previous section, we have seven linear equations in $P_1, P_2, P_3, Q_1, Q_2, Q_3$ and R , and the solution follows the pattern therein. The results for $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$ and $\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \lambda^{-1}$ are shown in Figs. 5 and 6, respectively, and are broadly similar to those shown in Figs. 1 and 3 except that the effect of B_1 is significantly stronger than that for B_2 . Indeed, much smaller values of B_1 than B_2 are required to produce comparable effects. The upper bound (7.68) depends on the magnitude B_1 but the values of ζ shown do not reflect this because of the relatively small values of B_1 used.

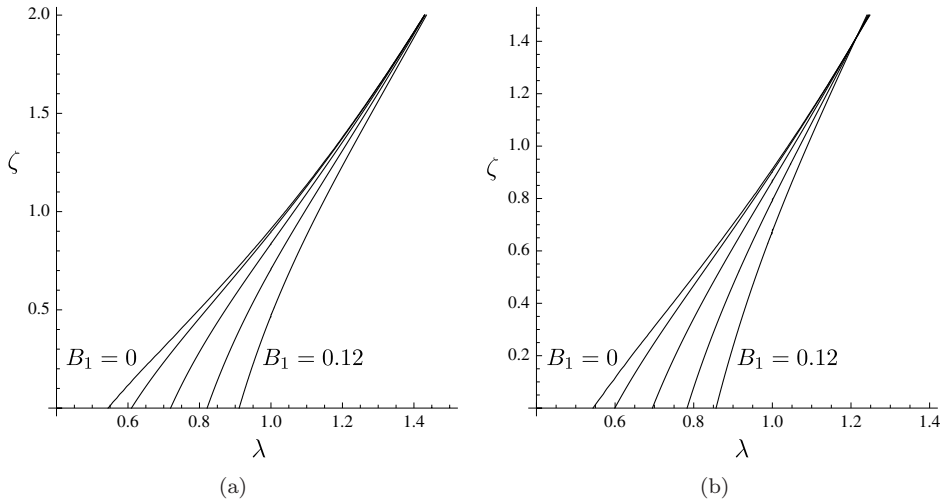


Fig. 5. Plot of $\zeta = \rho v^2 / \mu(0)$ versus $\lambda_1 = \lambda$ with $\lambda_3 = 1$ for $B_1 = 0, 0.03, 0.06, 0.09, 0.12$ T (curves reading from left to right): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 1, \mu_0 m = 1$.

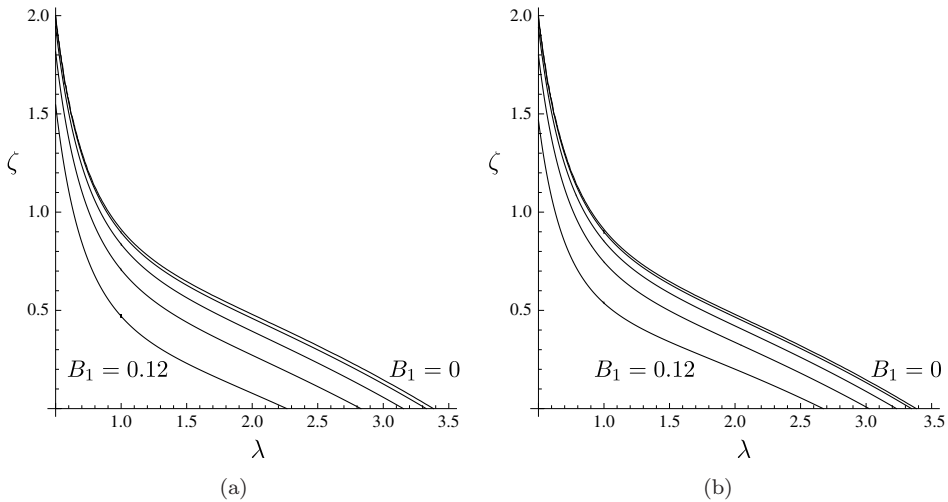


Fig. 6. Plot of $\zeta = \rho v^2 / \mu(0)$ versus $\lambda_2 = \lambda$ with $\lambda_1 = 1$ for $B_1 = 0, 0.03, 0.06, 0.09, 0.12$ T (curves reading from right to left): (a) $\mu_0 l = 2, \mu_0 m = 1$; (b) $\mu_0 l = 2, \mu_0 m = 0.2$.

8. Out-of-Plane Considerations

8.1. Magnetic induction components $(0, 0, B_3)$

The initial and deformed configurations are considered to be the same as in Sec. 7.2 except that the magnetic induction is taken to have components $(0, 0, B_3)$. The incremental quantities are as in the previous sections, i.e., we consider only incremental motions and magnetic induction components within the $(1, 2)$ plane. In fact,

the full three-dimensional (3D) equations decouple in this case and the out-of-plane motion can be considered separately, as discussed in Maugin and Hakmi [1995].

From Eqs. (6.8) and (6.9), with the components of the moduli tensors appropriately specialized, we obtain

$$\alpha\phi_{,1111} + 2\beta\phi_{,1122} + \gamma\phi_{,2222} = \rho(\phi_{,11} + \phi_{,22})_{,tt}, \tag{8.1}$$

$$K_{022}\psi_{,11} + K_{011}\psi_{,22} = 0, \tag{8.2}$$

which apply in $x_2 < 0$, and again (6.13) holds in $x_2 > 0$.

The boundary conditions for the underlying configuration require that $H_3^* = H_3$. Thus, $B_3^* = \mu_0 H_3 = 2\mu_0(l\lambda_3^{-2} + m)B_3$. If we assume there are no mechanical tractions, then $\tau_{22} = \tau_{22}^*$. The normal components of the Maxwell stress are $\tau_{22}^*(1, 1, -1)$, where $\tau_{22}^* = -B_3^{*2}/2\mu_0$. The incremental boundary conditions reduce to $\dot{T}_{021} = -\tau_{22}^*u_{2,1}$, $\dot{T}_{022} = -\tau_{22}^*u_{2,2}$, $\dot{T}_{023} = \mu_0^{-1}B_3^*\dot{B}_2^*$, $\dot{B}_{102} = \dot{B}_2^*$ and $\dot{H}_{101} = \dot{H}_1^*$. Note, in particular, the appearance of the out-of-plane shear traction term. After differentiating the \dot{T}_{022} condition with respect to x_1 , substituting for $\dot{p}_{,1}$ from an appropriately specialized form of (6.2) and then substituting for the potentials ϕ , ψ and ψ^* , we obtain (on dropping the factor $\gamma \neq 0$ from the first equation)

$$\phi_{,11} - \phi_{,22} = 0, \quad (2\beta + \gamma)\phi_{,112} + \gamma\phi_{,222} - \rho\phi_{,2tt} = 0 \quad \text{on } x_2 = 0, \tag{8.3}$$

$$m\psi_{,1} = (l + m\lambda_3^{-2})\psi_{,1}^*, \quad \psi_{,1} = \psi_{,1}^*, \quad K_{011}\psi_{,2} = \mu_0^{-1}\psi_{,2}^* \quad \text{on } x_2 = 0. \tag{8.4}$$

Except in the very special case for which $l = 0$ and $\lambda_3 = 1$ the latter equations are incompatible unless there is no incremental magnetic field. Thus, the problem reduces to a purely mechanical problem for the potential ϕ . For the considered model none of the moduli components depend on B_3 , so the magnetic field has no effect on the propagation of elastic surface waves. More generally, however, for the considered underlying deformation and magnetic field, Eq. (8.1) and the boundary conditions (8.3) apply for an arbitrary form of isotropic energy function Ω and therefore, the coefficients then do involve B_3 .

9. Concluding Remarks

The analysis in Sec. 7 shows that magnetic fields can have a significant effect on the speed of surface waves propagating in a half-space of magnetoelastic material and on the mechanical stability of the half-space. For each of the in-plane directions of the magnetic field an upper limit on the wave speed is obtained, similar to that obtained in the purely elastic case but with, in general, dependence on the magnetic field. In the absence of a magnetic field, the equations reduce to those of the purely elastic case given by Dowaiikh and Ogden [1990], and for the purely static problem results on the stability of a magnetoelastic half-space due to Otténio *et al.* [2008] are recovered.

For a Mooney–Rivlin type magnetoelastic material an initial magnetic induction in the sagittal plane in general destabilizes the material and surface waves exist

only for values of the stretch beyond a certain critical value (which depends on the chosen material parameters). If the magnetic induction is in the direction of wave propagation, it has a significantly stronger effect than in the case when it is perpendicular to the direction of wave propagation within the sagittal plane. For configurations in which the half-space is stable the dependence of the surface wave speed on both the underlying finite deformation and the magnitude of the magnetic induction was illustrated graphically.

We have also discussed briefly the equations governing in-plane motion for the situation in which the initial magnetic induction is normal to the sagittal plane. For the Mooney–Rivlin model it was found that the magnetic field has no effect on the surface wave speed. As is well known, the fully 3D equations decouple into planar and out-of-plane modes. We have not considered the out-of-plane (SH or Bleustein–Gulyaev wave) motion here but the combined effect of deformation and the magnetic field on such motions will be examined in a forthcoming paper.

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Appendix

A. Derivatives of the Invariants

The first and second derivatives of the invariants (4.4) and (4.5) with respect to \mathbf{F} and \mathbf{B}_l were given in Otténio *et al.* [2008]. We repeat the nonzero ones here for ease of reference.

$$\begin{aligned} \frac{\partial I_1}{\partial F_{i\alpha}} &= 2F_{i\alpha}, & \frac{\partial I_2}{\partial F_{i\alpha}} &= 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), & \frac{\partial I_3}{\partial F_{i\alpha}} &= 2I_3F_{\alpha i}^{-1}, \\ \frac{\partial I_5}{\partial F_{i\alpha}} &= 2B_{l\alpha}(F_{i\gamma}B_{l\gamma}), & \frac{\partial I_6}{\partial F_{i\alpha}} &= 2(F_{i\gamma}B_{l\gamma}c_{\alpha\beta}B_{l\beta} + F_{i\gamma}c_{\gamma\beta}B_{l\beta}B_{l\alpha}), \\ \frac{\partial I_4}{\partial B_{l\alpha}} &= 2B_{l\alpha}, & \frac{\partial I_5}{\partial B_{l\alpha}} &= 2c_{\alpha\beta}B_{l\beta}, & \frac{\partial I_6}{\partial B_{l\alpha}} &= 2c_{\alpha\gamma}c_{\gamma\beta}B_{l\beta}, & \frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2\delta_{ij}\delta_{\alpha\beta}, \\ \frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}), \\ \frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} &= 4I_3F_{\alpha i}^{-1}F_{\beta j}^{-1} - 2I_3F_{\alpha j}^{-1}F_{\beta i}^{-1}, & \frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2\delta_{ij}B_{l\alpha}B_{l\beta}, \\ \frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} &= 2[\delta_{ij}(c_{\alpha\gamma}B_{l\gamma}B_{l\beta} + c_{\beta\gamma}B_{l\gamma}B_{l\alpha}) + \delta_{\alpha\beta}F_{i\gamma}B_{l\gamma}F_{j\delta}B_{l\delta} \\ &\quad + F_{i\gamma}B_{l\gamma}F_{j\alpha}B_{l\beta} + F_{j\gamma}B_{l\gamma}F_{i\beta}B_{l\alpha} + b_{ij}B_{l\alpha}B_{l\beta}], \end{aligned}$$

$$\frac{\partial^2 I_5}{\partial F_{i\alpha} \partial B_{l\beta}} = 2\delta_{\alpha\beta} F_{i\gamma} B_{l\gamma} + 2B_{l\alpha} F_{i\beta},$$

$$\frac{\partial^2 I_6}{\partial F_{i\alpha} \partial B_{l\beta}} = 2F_{i\beta} c_{\alpha\gamma} B_{l\gamma} + 2F_{i\gamma} B_{l\gamma} c_{\alpha\beta} + 2F_{i\gamma} c_{\gamma\beta} B_{l\alpha} + 2\delta_{\alpha\beta} F_{i\gamma} c_{\gamma\delta} B_{l\delta},$$

$$\frac{\partial^2 I_4}{\partial B_{l\alpha} \partial B_{l\beta}} = 2\delta_{\alpha\beta}, \quad \frac{\partial^2 I_5}{\partial B_{l\alpha} \partial B_{l\beta}} = 2c_{\alpha\beta}, \quad \frac{\partial^2 I_6}{\partial B_{l\alpha} \partial B_{l\beta}} = 2c_{\alpha\gamma} c_{\gamma\beta}.$$

B. Magnetoelastic Tensors

For an isotropic material, $\mathbf{A}_0, \mathbf{C}_0$ and \mathbf{K}_0 can be expanded in terms of the derivatives of the invariants as follows, with $\Omega_n = \partial\Omega/\partial I_n$ and $\Omega_{mn} = \partial^2\Omega/\partial I_m \partial I_n$:

$$\mathcal{A}_{0piqj} = J^{-1} \sum_{m \in \mathcal{I}} \sum_{n \in \mathcal{I}} \Omega_{mn} F_{p\alpha} F_{q\beta} \frac{\partial I_n}{\partial F_{i\alpha}} \frac{\partial I_m}{\partial F_{j\beta}} + J^{-1} \sum_{n \in \mathcal{I}} \Omega_n F_{p\alpha} F_{q\beta} \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial F_{j\beta}},$$

$$\mathcal{C}_{0jilk} = \sum_{m \in \mathcal{J}} \sum_{n \in \mathcal{I}} \Omega_{mn} F_{j\alpha} F_{\beta k}^{-1} \frac{\partial I_m}{\partial B_{l\beta}} \frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^6 \Omega_n F_{j\alpha} F_{\beta k}^{-1} \frac{\partial^2 I_n}{\partial F_{i\alpha} \partial B_{l\beta}},$$

$$\mathbf{K}_{0ij} = J \sum_{m \in \mathcal{J}} \sum_{n \in \mathcal{J}} \Omega_{mn} F_{\alpha i}^{-1} F_{\beta j} \frac{\partial I_m}{\partial B_{l\alpha}} \frac{\partial I_n}{\partial B_{l\beta}} + J \sum_{n \in \mathcal{J}} \Omega_n F_{\alpha i}^{-1} F_{\beta j} \frac{\partial^2 I_n}{\partial B_{l\alpha} \partial B_{l\beta}}.$$

We recall that $\mathcal{I} = \{1, 2, 3, 5, 6\}$ and $\mathcal{J} = \{4, 5, 6\}$. For an incompressible material $\mathcal{I} = \{1, 2, 5, 6\}$ and $J = 1$.

When referred to the principal axes of the left Cauchy–Green tensor \mathbf{b} with principal stretches $\lambda_1, \lambda_2, \lambda_3$ and components (B_1, B_2, B_3) of the magnetic induction \mathbf{B} the components of $\mathbf{A}_0, \mathbf{C}_0$ and \mathbf{K}_0 are given explicitly for a compressible material as, for $i \neq j \neq k \neq i$,

$$\begin{aligned} \mathcal{A}_{0iiii} &= 2J^{-1} \lambda_i^2 [\Omega_1 + (\lambda_j^2 + \lambda_k^2) \Omega_2 + \lambda_j^2 \lambda_k^2 \Omega_3 + \lambda_j^2 \lambda_k^2 B_i^2 (\Omega_5 + 6\lambda_i^2 \Omega_6)] \\ &\quad + 4J^{-1} \lambda_i^4 \{ \Omega_{11} + 2(\lambda_j^2 + \lambda_k^2) \Omega_{12} + (\lambda_j^2 + \lambda_k^2)^2 \Omega_{22} \\ &\quad + \lambda_j^2 \lambda_k^2 [2\Omega_{13} + 2(\lambda_j^2 + \lambda_k^2) \Omega_{23} + \lambda_j^2 \lambda_k^2 \Omega_{33}] + 2\lambda_j^2 \lambda_k^2 B_i^2 [\Omega_{15} + 2\lambda_i^2 \Omega_{16} \\ &\quad + (\lambda_j^2 + \lambda_k^2) \Omega_{25} + 2\lambda_i^2 (\lambda_j^2 + \lambda_k^2) \Omega_{26} + \lambda_j^2 \lambda_k^2 \Omega_{35} + 2I_3 \Omega_{36}] \\ &\quad + \lambda_j^4 \lambda_k^4 B_i^4 (\Omega_{55} + 4\lambda_i^2 \Omega_{56} + 4\lambda_i^4 \Omega_{66}) \}, \\ \mathcal{A}_{0iiij} &= 4B_i B_j J \lambda_i^2 \{ \Omega_6 + \Omega_{15} + (\lambda_j^2 + \lambda_k^2) \Omega_{25} + \lambda_j^2 \lambda_k^2 \Omega_{35} \\ &\quad + (\lambda_i^2 + \lambda_j^2) [\Omega_{16} + (\lambda_j^2 + \lambda_k^2) \Omega_{26} + \lambda_j^2 \lambda_k^2 \Omega_{36}] \\ &\quad + \lambda_j^2 \lambda_k^2 B_i^2 [\Omega_{55} + (3\lambda_i^2 + \lambda_j^2) \Omega_{56} + 2\lambda_i^2 (\lambda_i^2 + \lambda_j^2) \Omega_{66}] \}, \\ \mathcal{A}_{0iiji} &= 2B_i B_j J \{ \Omega_5 + (\lambda_j^2 + 3\lambda_i^2) \Omega_6 + 2\lambda_i^2 [\Omega_{15} + (\lambda_j^2 + \lambda_k^2) \Omega_{25} + \lambda_j^2 \lambda_k^2 \Omega_{35}] \\ &\quad + 2\lambda_i^2 (\lambda_i^2 + \lambda_j^2) [\Omega_{16} + (\lambda_j^2 + \lambda_k^2) \Omega_{26} + \lambda_j^2 \lambda_k^2 \Omega_{36}] \\ &\quad + 2J^2 B_i^2 [\Omega_{55} + (3\lambda_i^2 + \lambda_j^2) \Omega_{56} + 2\lambda_i^2 (\lambda_i^2 + \lambda_j^2) \Omega_{66}] \}, \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_{0iijj} &= 4J^{-1}\lambda_i^2\lambda_j^2\{\Omega_2 + \lambda_k^2\Omega_3 + \Omega_{11} + (I_1 + \lambda_k^2)\Omega_{12} + (I_2 + \lambda_k^4)\Omega_{22} \\
 &\quad + \lambda_k^2[(\lambda_i^2 + \lambda_j^2)\Omega_{13} + (I_2 + \lambda_i^2\lambda_j^2)\Omega_{23} + I_3\Omega_{33}] \\
 &\quad + \lambda_k^2(\lambda_j^2B_i^2 + \lambda_i^2B_j^2)(\Omega_{15} + \lambda_k^2\Omega_{25}) + 2I_3(\lambda_i^2B_i^2 + \lambda_j^2B_j^2)(\Omega_{26} + \lambda_k^2\Omega_{36}) \\
 &\quad + I_3(B_i^2 + B_j^2)(2\Omega_{16} + \Omega_{25} + 2\lambda_k^2\Omega_{26} + \lambda_k^2\Omega_{35}) \\
 &\quad + I_3\lambda_k^2B_i^2B_j^2[\Omega_{55} + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56} + 4\lambda_i^2\lambda_j^2\Omega_{66}]\}, \\
 \mathcal{A}_{0ijij} &= 2J^{-1}\lambda_i^2\{\Omega_1 + \lambda_k^2\Omega_2 + B_i^2\lambda_j^2\lambda_k^2\Omega_5 + \lambda_j^2\lambda_k^2(2B_i^2\lambda_i^2 + B_i^2\lambda_j^2 + B_j^2\lambda_i^2)\Omega_6 \\
 &\quad + 2B_i^2B_j^2J^2\lambda_j^2\lambda_k^2[\Omega_{55} + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56} + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}]\}, \\
 \mathcal{A}_{0ijji} &= 2J^{-1}\lambda_i^2\lambda_j^2\{-\Omega_2 - \lambda_k^2\Omega_3 + \lambda_k^2(\lambda_j^2B_i^2 + \lambda_i^2B_j^2)\Omega_6 \\
 &\quad + 2B_i^2B_j^2J^2\lambda_k^2[\Omega_{55} + 2(\lambda_i^2 + \lambda_j^2)\Omega_{56} + (\lambda_i^2 + \lambda_j^2)^2\Omega_{66}]\}, \\
 \mathcal{A}_{0iijk} &= 4B_jB_kJ\lambda_i^2\{\Omega_{15} + (\lambda_j^2 + \lambda_k^2)(\Omega_{25} + \Omega_{16}) + (\lambda_j^2 + \lambda_k^2)^2\Omega_{26} + \lambda_j^2\lambda_k^2\Omega_{35} \\
 &\quad + \lambda_j^2\lambda_k^2(\lambda_j^2 + \lambda_k^2)\Omega_{36} + B_i^2\lambda_j^2\lambda_k^2[\Omega_{55} + (I_1 + \lambda_i^2)\Omega_{56} + 2\lambda_i^2(\lambda_j^2 + \lambda_k^2)\Omega_{66}]\}, \\
 \mathcal{A}_{0ijk i} &= \mathcal{A}_{0ij k i} = 2B_jB_kJ\{\lambda_i^2\Omega_6 + 2B_i^2J^2[\Omega_{55} + (I_1 + \lambda_i^2)\Omega_{56} + (I_2 + \lambda_i^4)\Omega_{66}]\}, \\
 \mathcal{A}_{0jik i} &= 2B_jB_kJ\{\Omega_5 + I_1\Omega_6 + 2B_i^2J^2[\Omega_{55} + (I_1 + \lambda_i^2)\Omega_{56} + (I_2 + \lambda_i^4)\Omega_{66}]\}, \\
 \mathcal{C}_{0ii|i} &= 4B_iJ\{\Omega_5 + 2\lambda_i^2\Omega_6 + \Omega_{14} + \lambda_i^2\Omega_{15} + \lambda_i^4\Omega_{16} \\
 &\quad + (\lambda_j^2 + \lambda_k^2)(\Omega_{24} + \lambda_i^2\Omega_{25} + \lambda_i^4\Omega_{26}) + \lambda_j^2\lambda_k^2(\Omega_{34} + \lambda_i^2\Omega_{35} + \lambda_i^4\Omega_{36}) \\
 &\quad + B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56} + 2\lambda_i^2(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\}, \\
 \mathcal{C}_{0ii|j} &= 4B_jJ\lambda_i^2\lambda_j^{-2}\{\Omega_{14} + \lambda_j^2\Omega_{15} + \lambda_j^4\Omega_{16} + (\lambda_j^2 + \lambda_k^2)(\Omega_{24} + \lambda_j^2\Omega_{25} + \lambda_j^4\Omega_{26}) \\
 &\quad + \lambda_j^2\lambda_k^2(\Omega_{34} + \lambda_j^2\Omega_{35} + \lambda_j^4\Omega_{36}) + B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_j^2\Omega_{55} + \lambda_j^4\Omega_{56} \\
 &\quad + 2\lambda_j^2(\Omega_{46} + \lambda_j^2\Omega_{56} + \lambda_j^4\Omega_{66})]\}, \\
 \mathcal{C}_{0ij|i} &= 2B_jJ\{\Omega_5 + (\lambda_i^2 + \lambda_j^2)\Omega_6 + 2B_i^2\lambda_j^2\lambda_k^2[\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56} \\
 &\quad + (\lambda_i^2 + \lambda_j^2)(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\}, \\
 \mathcal{C}_{0ij|k} &= 4B_iB_jB_kJ\lambda_i^2\lambda_j^2[\Omega_{45} + \lambda_k^2\Omega_{55} + \lambda_k^4\Omega_{56} + (\lambda_i^2 + \lambda_j^2)(\Omega_{46} + \lambda_k^2\Omega_{56} + \lambda_k^4\Omega_{66})], \\
 \mathcal{K}_{0ii} &= 2J\lambda_i^{-2}\{\Omega_4 + \lambda_i^2\Omega_5 + \lambda_i^4\Omega_6 + 2B_i^2\lambda_j^2\lambda_k^2[\Omega_{44} + \lambda_i^2\Omega_{45} + \lambda_i^4\Omega_{46} \\
 &\quad + \lambda_i^2(\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56}) + \lambda_i^4(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})]\}, \\
 \mathcal{K}_{0ij} &= 4B_iB_jJ\lambda_k^2[\Omega_{44} + \lambda_i^2\Omega_{45} + \lambda_i^4\Omega_{46} + \lambda_j^2(\Omega_{45} + \lambda_i^2\Omega_{55} + \lambda_i^4\Omega_{56}) \\
 &\quad + \lambda_j^4(\Omega_{46} + \lambda_i^2\Omega_{56} + \lambda_i^4\Omega_{66})].
 \end{aligned}$$

For an incompressible material the above formulas apply with $J = 1, I_3 = 1$ and with all terms in Ω carrying a subscript 3 omitted.

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