A 2D EULER-BERNOULLI INELASTIC BEAM-COLUMN ELEMENT FOR THE LARGE INCREMENT METHOD

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ABSTRACT

The present work is part of a research on modelling a forced-based layered beam-column element for use with the Large Increment Method (LIM). The aim is to investigate the accuracy and robustness of this approach to modelling the general inelastic behaviour of strain-hardening materials for an Euler-Bernoulli beam element using general inverse matrix theory. In this paper, the beam-column element variables are defined, and the efficiency and accuracy of the element in modelling three benchmark example problems is assessed.

1. INTRODUCTION

When employing the finite element method the behavior of structures under arbitrary load can be analyzed using either a displacement-based, a force-based or a mixed numerical formulation. The optimum choice depends on the method’s accuracy, rate of convergence, applicability to various types of finite elements involved in the discretisation process, its ability to pass over critical or limit points where load or displacement is at a maximum, as well as on mesh size independence [1,2].

In the force based category, one possible solution algorithm is the Large Increment Method (LIM), which is fundamentally different from other force-based solution methods such as the Integrated Force Method (IFM) and the LArge Time Increment (LATIN), presented by Patnaik and Ladeveze[3,4,5].

So far, in contrast to the LIM, other force based methods and displacement based methods follow a linearized process. Therefore, error accumulation is an unavoidable consequence of these techniques [3,6]. Since in the LIM, the equilibrium and compatibility equations are associated with the entire structural system, whilst the constitutive equations are independent for each element, the constitutive equations can be considered at the local elemental level in the solution process, in contrast to the other approaches, where it is considered at the global level of the solution process. This represents an important advantage of the LIM, as it avoids the need for linearization, which makes the LIM a good candidate for inelastic analysis. The LIM can thus be considered to be a good numerical procedure due to the following reasons; (more details can be found in [3]);

☐ The equilibrium equations are satisfied at all times,
☐ The equilibrium and compatibility equations are considered in a single step, effectively decreasing calculation time,
☐ Linearization is avoided, eliminating accumulated error,
☐ Fewer elements for a given accuracy are required, compared to the displacement-based method [6].
2. DERIVATION OF THE LIM GOVERNING EQUATIONS

Let the structure domain be denoted Ω, the force and displacement boundaries be ∂Ωe, ∂Ωg where, ∂Ωe ∩ ∂Ωg = 0, ∂Ωe ∪ ∂Ωg = ∂Ω. For this domain, the three main structural analysis equations are; the equilibrium equations \( \sigma_{ij} + b_i(\Omega) = 0 \), the general constitutive relationship \( \varepsilon_{ij} = \mathbf{\Pi}(\sigma_{ij}, \varepsilon_{ij}^p) \) and the compatibility equations \( \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \).

The generalized element forces are given as, \( F = f_i, i = 1,..,n \) and nodal displacements as, \( D = D_i, i = 1,..,m \) for discrete elements. Based on the principle of virtual work

\[
\int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega = \int_{\partial\Omega_e} t_i \, d\Gamma + \int_{\partial\Omega_g} t_i (B) \, d\Gamma \quad \text{where,} \quad t_i (B) = \sigma_{ij} n_j u_i = u_i(B) \quad \text{on related boundaries}
\]

\( \varepsilon_{ij} = B_{ik} d_k \), the LIM equilibrium equation can be written as \( CF = P \) where \( C = \int B^T Z d\Omega \), and

\[
P = \int_{\Omega^e} N^T t(B) dS + \int_{\Omega^p} N^T f(\Omega) d\Omega \quad \text{in both the elastic and plastic domains. Here } Z \text{ is the shape function in the force based method and } N, B \text{ are the shape function and the strain-displacement matrix respectively. Eventually, in a general structure with } n \text{ element DOFs and } m \text{ nodal DOFs the equilibrium equation can be written as } C_{mn} F_{mn} = P_{mn}. \text{ Based on the General Inverse method } (C^{-1}_{mn} = C_{mn} (C_{mn} C_{mn}^T)^{-1}) \text{ the modified form is } F = C^{-1}_{mn} P + \beta_{mn} X \text{ where } \beta = I_{mn} - \alpha, \alpha = C^{-1}_{mn} C, \text{ and the conjugate gradient } (\beta_{mn} X) \text{ for the residual force, based on unbalanced deformation, have been defined in previous research [3, 7].}

Using the principle of the complementary virtual work ( \( \int_{\Omega^e+\Omega^p} (\varepsilon_{ij} - B_{ik} d_k) \, d\Omega = 0 \) ) the LIM compatibility equation for the system is \( C^T D = \delta \) where \( \delta = \int_{\Omega^e+\Omega^p} Z^T \varepsilon \, d\Omega \). This equation can be written as; \( \alpha \delta = \alpha C^T D = \alpha \left( C^T \left( C C^T \right)^{-1} \right) C^T D \) or \( (I - \alpha) \delta = 0 \). Thus, the condition \( \beta \delta = 0 \) is a criterion that has to be met, in order to satisfy the compatibility equations.

Finally, the constitutive equation at the local level can be written as \( \delta = \tilde{f}(F, \delta^p) \) for each element.

![Fig.1 Element configuration](image)

3. STATE DETERMINATION FOR THE BEAM-COLUMN ELEMENT

One of the most common cases considered when modelling inelastic behaviour of frame-like structures is to restrict inelasticity to a formation of hinges at both ends of a beam element. To adapt this method for the LIM an effective symmetric formulation with rotational and axial deformations based on Euler-Bernoulli beam theory is proposed with 6 nodal DOFs and 3 element DOFs: \( \tilde{f} = f_i, i = 1,..,6 \),
\( f = f_i, i = 1, 2, 3 \) in 2D where the matrix \( T \) is defined as the element equilibrium matrix (see Fig.1). Therefore, the stress at any point along the beam length and depth, in both the elastic and inelastic regions, can be expressed as a function of the element force, \( \sigma = ZF \) where \( (F_i = \{f_1, f_2, f_3\}^T, Z_i = \{Z_1, Z_2, Z_3\}) \), and the deformation can be calculated using \( \delta_i = \int Z_i \sigma \, d\Omega \). As long as the strain is less than the yielding strain (\( \varepsilon_y \)) the material behaviour is elastic, therefore the stress in this region can be calculated very simply as, \( Z = \{(x - L), x, LI / yA\}/LI \). For strain greater than \( \varepsilon_y \), the behaviour is inelastic and the stress is not linear. When the strain is less than the yielding strain \( Z = \{(x - L)\sigma_x, x\sigma_x, LI_yM_y / yA\}/LIyM_y, \) otherwise the stress depends on the material inelastic behaviour defined by, \( Z = \{(x - L)\sigma(y), x\sigma(y), LI_M / A\}/LI_y \). It should be mentioned that for a general section, the neutral plane location \( (Y_{N,A}) \) and elastic depth \( (Y_p) \) are calculated from \( N = \int_A \sigma \, dA \), \( M = \int_A \sigma y \, dA \). For the elastic perfectly plastic behavior when considering a simple rectangular section, the equations are simple, but for a general section and more complex material behaviour, the equations become much more complicated. Thus, a simplification is necessary; by using an interaction force-moment curve to calculate the strain distribution over the entire section, a correct solution can be found. Although this method requires a large predetermined database of information to draw upon, it is a possible solution to devising for a general section. Eventually, the relation between the force and deformation and the moment vs rotation at both ends can be defined, where a calibration factor, \( \phi \), can be calculated for each element.

\[
\begin{bmatrix}
K_1 - \frac{K_2^2}{K_1} (1 - \phi) & \phi_d \phi_1 C & 0 \\
\phi_d \phi_2 C & K_1 - \frac{K_2^2}{K_1} (1 - \phi) & 0 \\
0 & 0 & AE/L
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_d
\end{bmatrix}
= \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\Delta
\end{bmatrix}
= \begin{bmatrix}
M_1 \\
M_2 \\
N
\end{bmatrix},
K_1 = 2K_2 = 4EI/L
\]

### 4. Numerical Examples

The following numerical examples are analyzed and compared against results produced by the commercial finite element code, ABAQUS™ in order to demonstrate the convergence and robustness of the code for inelastic behavior.

Example I. A simple column, subjected to an axial force with nonlinear material behavior is solved with 4 iterations. The small differences between results of the LIM code and the commercial code for both displacement and force predictions, demonstrates the LIM’s accuracy, see Fig. 2

<table>
<thead>
<tr>
<th>i</th>
<th>Force</th>
<th>Difference ( ( % ) )</th>
<th>Displacement</th>
<th>Difference ( ( % ) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>-9.195947</td>
<td>0.1005</td>
<td>0.02958452</td>
</tr>
<tr>
<td>2</td>
<td>5.5</td>
<td>-0.115542</td>
<td>0.09749</td>
<td>0.00002476</td>
</tr>
<tr>
<td>3</td>
<td>5.5063452</td>
<td>-0.0000308</td>
<td>0.0974645863409050200</td>
<td>0.00000064</td>
</tr>
<tr>
<td>4</td>
<td>5.5063621</td>
<td>-0.000001</td>
<td>0.09746458624728530</td>
<td>-0.00000001</td>
</tr>
</tbody>
</table>

Fig. 2. Comparison of LIM and ABAQUS™ results for example I

Example II. A truss structure subjected to a cyclic load, with six elements and four nodal DOFs with bilinear material behavior, is analyzed. The load-displacement diagram at node 1, for isotropic and kinematics behavior is presented in Fig.3
Fig. 3. Comparison of LIM and ABAQUS result for example II

Example III. A continuous beam, subjected to cyclic moments and axial force made of bilinear material is analyzed. The moment-rotation diagram at node 2 is presented in Fig. 4. The end deformation based on maximum strain (Method I), strain in inelastic region (Method II) and whole of the element (Method III) are calculated using both 1D and 3D inelastic behaviours.

Fig. 4. Comparison of LIM and ABAQUS result for example III

5. CONCLUSIONS

In this paper, a force-based layered Euler-Bernoulli element enabling inelastic behavior to be treated using the LIM is derived and implemented. The accuracy of this method is illustrated by comparing predictions of the new element with those from the commercial finite element code, ABAQUS. This element can be extended for more general behaviour (e.g., out of plane bending, shear and torsion). Further results will be published in the near future.

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References